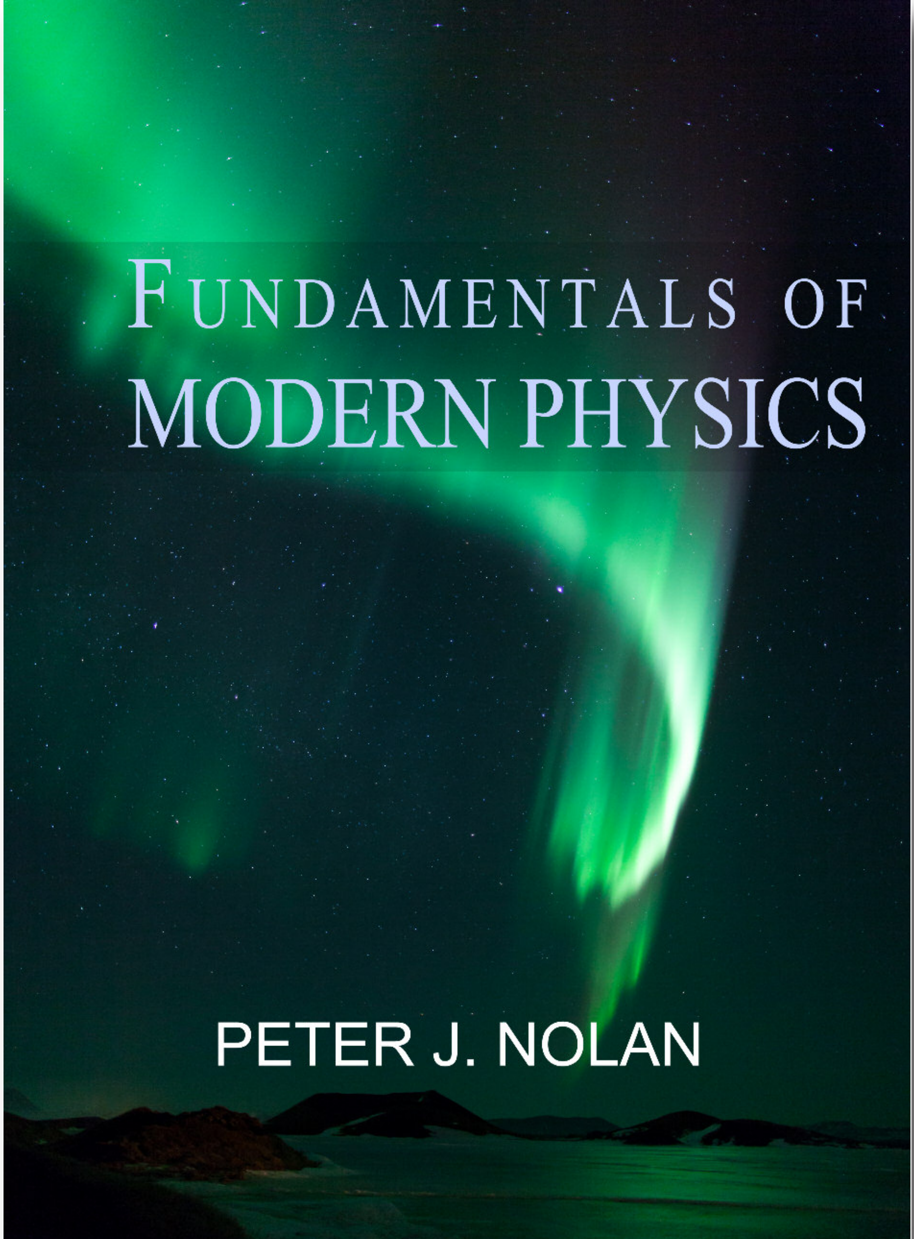

FUNDAMENTALS OF MODERN PHYSICS

First Edition

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Contents

This book is dedicated to my family—
my wife Barbara,
my sons,
Thomas, James, John and Kevin,
my daughters' in-law,
Joanne and Nancy,
my grandchildren,
Joseph, Kathleen, Shannon, and Erin.

Preface

This text gives a good, traditional coverage for students of Modern Physics. The organization of the text follows the traditional sequence of Special Relativity, General Relativity, Quantum Physics, Atomic Physics, Nuclear Physics, and Elementary Particle Physics and the Unification of the Forces. The emphasis throughout the book is on simplicity and clarity.

There are a large number of diagrams and illustrative problems in the text to help students visualize physical ideas. Important equations are highlighted to help students find and recognize them. A summary of these important equations is given at the end of each chapter.

To simplify the learning process, every illustrative example in the textbook is linked to an Excel spreadsheet (Microsoft Excel must be installed on the computer). These *Interactive Examples* will allow the student to solve the example problem in the textbook, with all the in-between steps, many times over but with different numbers placed in the problem. More details on these *Interactive Examples* can be found in the section "Interactive Examples with Excel" at the end of the Preface.

Students sometimes have difficulty remembering the meanings of all the vocabulary associated with new physical ideas. Therefore, a section called *The Language of Physics*, found at the end of each chapter, contains the most important ideas and definitions discussed in that chapter.

To comprehend the physical ideas expressed in the theory class, students need to be able to solve problems for themselves. Problem sets at the end of each chapter are grouped according to the section where the topic is covered. Problems that are a mix of different sections are found in the Additional Problems section. If you have difficulty with a problem, refer to that section of the chapter for help. The problems begin with simple, plug-in problems to develop students' confidence and to give them a feel for the numerical magnitudes of some physical quantities. The problems then become progressively more difficult and end with some that are very challenging. The more difficult problems are indicated by a star (*). The starred problems are either conceptually more difficult or very long. Many problems at the end of the chapter are very similar to the illustrative problems worked out in the text. When solving these problems, students can use the illustrative problems as a guide, and use the *Interactive Examples* as a check on their work.

A section called *Interactive Tutorials*, which also uses Excel spreadsheets to solve physics problems, can be found at the end of the problems section in each chapter. These *Interactive Tutorials* are a series of problems, very much like the *Interactive Examples*, but are more detailed and more general. More details on these *Interactive Tutorials* can be found in the section "Interactive Tutorials with Excel" at the end of the Preface.

A series of questions relating to the topics discussed in the chapter is also included at the end of each chapter. Students should try to answer these questions to see if they fully understand the ramifications of the theory discussed in the chapter. Just as with the problem sets, some of these questions are either conceptually more difficult or will entail some outside reading. These more difficult questions are also indicated by a star (*).

In this book only SI units will be used in the description of physics. Occasionally, a few problems throughout the book will still have some numbers in the British Engineering System of Units. When this occurs the student should convert these numbers into SI units, and proceed in solving the problem in the International System of Units.

A Bibliography, given at the end of the book, lists some of the large number of books that are accessible to students taking modern physics. These books cover such topics in modern physics as relativity, quantum mechanics, and elementary particles. Although many of these books are of a popular nature, they do require some physics background. After finishing this book, students should be able to read any of them for pleasure without difficulty.

A Special Note to the Student

"One thing I have learned in a long life: that all our science measured against reality, is primitive and childlike--and yet it is the most precious thing we have."

Albert Einstein
as quoted by Banesh Hoffmann in
Albert Einstein, Creator and Rebel

The language of physics is mathematics, so it is necessary to use mathematics in our study of nature. However, just as sometimes "you cannot see the forest for the trees," you must be careful or "you will not see the physics for the mathematics." Remember, mathematics is only a tool used to help describe the physical world. You must be careful to avoid getting lost in the mathematics and thereby losing sight of the physics. When solving problems, a sketch or diagram that represents the physics of the problem should be drawn first, then the mathematics should be added.

Physics is such a logical subject that when a student sees an illustrative problem worked out, either in the textbook or on the blackboard, it usually seems very simple. Unfortunately, for most students, it is simple only until they sit down and try to do a problem on their own. Then they often find themselves confused and frustrated because they do not know how to get started.

If this happens to you, do not feel discouraged. It is a normal phenomenon that happens to many students. The usual approach to overcoming this difficulty is going back to the illustrative problem in the text. When you do so, however, do not look at the solution of the problem first. Read the problem carefully, and then try to solve the problem on your own. At any point in the solution, when you cannot proceed to the next step on your own, peek at that step and only that step in the illustrative problem. The illustrative problem shows you what to do at that step. Then continue to solve the problem on your own. Every time you get stuck, look again at the appropriate solution step in the illustrative problem until you can finish the entire problem. The reason you had difficulty at a particular place in the problem is usually that you did not understand the physics at that point as well as you thought you did. It will help to reread the appropriate theory section. Getting stuck on a problem is not a bad thing, because each time you do, you have the opportunity to learn something. Getting stuck is the first step on the road to knowledge. I hope you will feel comforted to know that most of the students who have gone before you also had these difficulties. You are not alone. Just keep trying. Eventually, you will find that solving physics problems is not as difficult as you first thought; in fact, with time, you will find that they can even be fun to solve. The more problems that you solve, the easier they become, and the greater will be your enjoyment of the course.

Interactive Examples with Excel

The *Interactive Examples* in the book will allow the student to solve the example problem in the textbook, with all the in-between steps, many times over but with different numbers placed in the problem (Microsoft Excel must be installed on the computer). Figure 1 shows an example from Chapter 1 of the textbook for solving a problem dealing with the Lorentz contraction. It is a problem in special relativity in which a man on the earth measures an event at a particular point from him at a particular time. If a rocket ship flies over the man at a particular speed, what coordinates does the astronaut in the rocket ship attribute to this event?

The example in the textbook shows all the steps and reasoning done in the solution of the problem.

Example 1.5

Lorentz transformation of coordinates. A man on the earth measures an event at a point 5.00 m from him at a time of 3.00 s. If a rocket ship flies over the man at a speed of 0.800*c,* what coordinates does the astronaut in the rocket ship attribute to this event?

Solution

The location of the event, as observed in the moving rocket ship, found from equation 1.49, is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{5.00\ \text{m} - (0.800)(3.00\times10^8\ \text{m/s})(3.00\ \text{s})}{\sqrt{1-(0.800c)^2/c^2}}$$
$$= -1.20 \times 10^9\ \text{m}$$

This distance is quite large because the astronaut is moving at such high speed. The event occurs on the astronaut's clock at a time

$$t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}}$$
$$= \frac{3.00\ \text{s} - (0.800)(3.00\times10^8\ \text{m/s})(5.00\ \text{m})/(3.00\times10^8\ \text{m/s})^2}{\sqrt{1-(0.800c)^2/c^2}}$$
$$= 5.00\ \text{s}$$

Go to Interactive Example

Figure 1 Example 1.5 in the textbook.

The last sentence in blue type in the example allows the student to access the interactive example for this same problem. Clicking on the blue sentence opens the spreadsheet shown in figure 2. Notice that the problem is stated in the identical manner as in the textbook. Directly below the stated problem is a group of yellow-colored cells labeled *Initial Conditions*. Into these yellow cells are placed the numerical values associated with the particular problem. The problem is now solved in the identical way it is solved in the textbook. Words are used to describe the physical principles and then the equations are written down. Then the in-between steps of the calculation are shown in light green-colored cells, and the final result of the calculation is shown in a light blue-colored cell. The entire problem is solved in this manner, as shown in figure 2. If the student wishes to change the problem by using a different initial condition, he or she then changes these values in the yellow-colored cells of the initial conditions. When the initial conditions are changed the spreadsheet recalculates all the new in-between steps in the problem and all the new final answers to the problem. In this way the problem is completely interactive. It changes for every new set of initial conditions. *The Interactive Examples make the book a living book. The examples can be changed many times over to solve for all kinds of special cases.*

"The Fundamentals of the Theory of Modern Physics"

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Chapter 1 Special Relativity

Computer Assisted Instruction

Interactive Examples

Example 1.5

Lorentz transformation of coordinates. A man on the earth measures an event at a point 5.00 m from him at a time of 3.00 s. If a rocket ship flies over the man at a speed of 0.800c, what coordinates does the astronaut in the rocket ship attribute to this event?

Initial Conditions

x =	5	m				t =	3	s
v =	0.8	c	=	2.4E+08	m/s	c =	3.00E+08	m/s

Solution.

The location of the event, as observed in the moving rocket ship, found from equation 1.49, is

$$x' = (x - v\,t)/\mathrm{sqrt}[1 - v^2 / c^2]$$

x' = [(5 m) - (2.4E+08 m/s) x (3 s)]
/ sqrt[1 - (0.8 $c)^2$ / (1 $c)^2$}]
x' = -1.20E+09 m

This distance is quite large because the astronaut is moving at such high speed. The event occurs on the astronaut's clock at a time

$$t' = (t - v\,x / c^2) / \mathrm{sqrt}[1 - v^2 / c^2]$$

t' = [(3 s) - (2.4E+08 m/s) x (5 m) / (3.00E+08 $m/s)^2$]
/ sqrt[1 - (2.40E+08 $m/s)^2$ / (3.00E+08 $c)^2$}]
t' = 5 s

Figure 2 Interactive Example 1.5 in Microsoft Excel Spreadsheet.

These *Interactive Examples* are a very helpful tool to aid in the learning of modern physics if they are used properly. The student should try to solve the particular problem in the traditional way using paper and a calculator. Then the student should open the spreadsheet, insert the appropriate data into the Initial Conditions cells and see how the computer solves the problem. Go through each step on the computer and compare it to the steps you made on paper. Does your answer agree? If not, check through all the in-between steps on the spreadsheet and your paper and find where your made a mistake. *Do not feel bad if you make a mistake. There is nothing wrong in making a mistake, what is wrong is not learning from your mistake.* Now that you understand your mistake, repeat the problem using different Initial Conditions on the spreadsheet and your paper. Again check your answers and all the in-between steps. Once you are sure that you know how to solve the problem, try some special cases. What would happen if you changed an angle, a weight, a force? In this way you can get a great deal of insight into the physics of the problem and also learn a great deal of modern physics in the process.

You must be very careful not to just plug numbers into the Initial Conditions and look at the answers without understanding the in-between steps and the actual physics of the problem. You will only be deceiving yourself. Be careful, these spreadsheets can be extremely helpful if they are used properly.

We should point out two differences in a text example and in a spreadsheet example. Powers of ten that are used in scientific notation in the text are written with the capital letter E in the spreadsheet. Hence, the number 5280, written in scientific notation as 5.280×10^3, will be written on the spreadsheet as 5.280E+3. Also, the square root symbol, $\sqrt{\ }$, in the textbook is written as sqrt[ ] in a spreadsheet. Finally, we should note that the spreadsheets are "protected" by allowing you to enter data only in the designated light yellow-colored cells of the Initial Conditions area. Therefore, the student cannot damage the spreadsheets in any way, and they can be used over and over again.

Interactive Tutorials with Excel

Besides the *Interactive Examples* in this text, I have also introduced a section called *Interactive Tutorials* at the end of the problem section in each chapter. These Interactive Tutorials are a series of problems, very much like the *Interactive Examples*, but are more detailed and more general.

To access the Interactive Tutorial, the student will click on the sentence in blue type at the end of the Interactive Tutorials section. Clicking on the blue sentence opens the appropriate spreadsheet.

Figure 3 show a typical *Interactive Tutorial* for problem 46 in chapter 1. It shows the change in mass of an object when it's in motion. When the student opens this particular spreadsheet, he or she sees the problem stated in the usual manner.

"The Fundamentals of the Theory of Modern Physics"

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Chapter 1 Special Relativity

Computer Assisted Instruction

Interactive Tutorial

46. Relativistic mass. A mass at rest has a value m_o = 2.55 kg. Find the relativistic mass m when the object is moving at a speed v = 0.355 c.

Initial Conditions

m_o = 2.55 kg   c = 3.00E+08 m/s

v = 3.55E-01 c = 1.07E+08 m/s

For speeds that are not given in terms of the speed of light c use the following converter to find the equivalent speed in terms of the speed of light c. Then place the equivalent speed into the yellow cell for v above.

v = 1610 km/hr = 447.58 m/s = 1.49E-06 c

v = 1.61E+06 m/s = 5.37E-03 c

The relativistic mass is given by equation 1.86 as

$$m = m_o / \mathrm{sqrt}[1 - (v^2)/(c^2)]$$

m = (2.55 kg)/sqrt[1 - (1.07E+08 m/s)^2 / (3.00E+08 m/s)^2]

m = 2.7276628 kg

Figure 3 A typical Interactive Tutorial.

Directly below the stated problem is a group of yellow-colored cells labeled *Initial Conditions*.

Into these yellow cells are placed the numerical values associated with the particular problem. For this problem the initial conditions consist of the rest mass of the object, it speed, and the speed of light as shown in figure 3. The problem is now solved in the traditional way of a worked out example in the book. Words are used to describe the physical principles and then the equations are written down. Then the in-between steps of the calculation are shown in light green-colored cells, and the final result of the calculation is shown in a light blue-green-colored cell. The entire problem is solved in this manner as shown in figure 3. If the student wishes to change the problem by using a different initial mass or speed, he or she then changes these values in the yellowed-colored cells of the initial conditions. When the initial conditions are changed the spreadsheet recalculates all the new in-between steps in the problem and all the new final answers to the problem. In this way the problem is completely interactive. It changes for every new set of initial conditions. The tutorials can be changed many times over to solve for all kinds of special cases.

Chapter 1 Special Relativity

> And now in our time, there has been unloosed a cataclysm which has swept away space, time, and matter hitherto regarded as the firmest pillars of natural science, but only to make place for a view of things of wider scope, and entailing a deeper vision. This revolution was promoted essentially by the thought of one man, Albert Einstein.
>
> Hermann Weyl - Space-Time-Matter

1.1 Introduction to Relative Motion

Relativity *has as its basis the observation of the motion of a body by two different observers in relative motion to each other.* This observation, apparently innocuous when dealing with motions at low speeds has a revolutionary effect when the objects are moving at speeds near the velocity of light. At these high speeds, it becomes clear that the simple concepts of space and time studied in Newtonian physics no longer apply. Instead, *there becomes a fusion of space and time into one physical entity called spacetime. All physical events occur in the arena of spacetime. As we shall see, the normal Euclidean geometry, studied in high school, that applies to everyday objects in space does not apply to spacetime. That is, spacetime is non-Euclidean. The apparently strange effects of relativity, such as length contraction and time dilation, come as a result of this non-Euclidean geometry of spacetime.*

The earliest description of relative motion started with Aristotle who said that the earth was at absolute rest in the center of the universe and everything else moved relative to the earth. As a proof that the earth was at absolute rest, he reasoned that if you throw a rock straight upward it will fall back to the same place from which it was thrown. If the earth moved, then the rock would be displaced on landing by the amount that the earth moved. This is shown in figures 1.1(a) and 1.1(b).

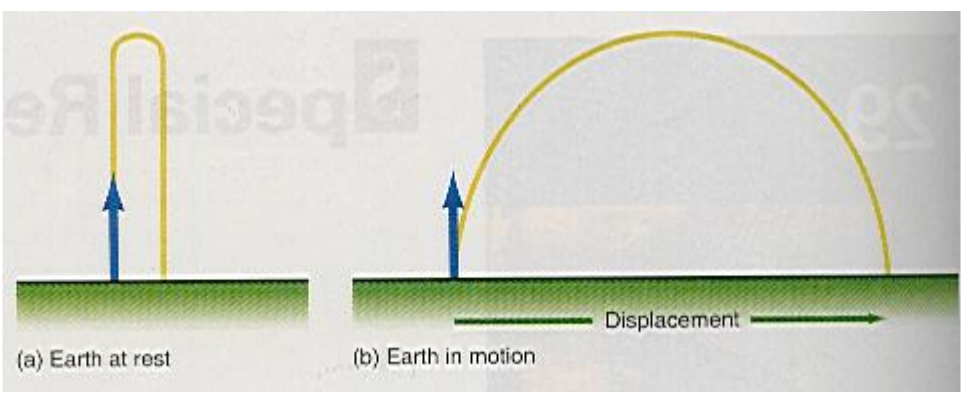


Figure 1.1 Aristotle's argument for the earth's being at rest.

Based on the prestige of Aristotle, the belief that the earth was at absolute rest was maintained until Galileo Galilee (1564-1642) pointed out the error in Aristotle's reasoning. Galileo suggested that if you throw a rock straight upward in a boat that is moving at constant velocity, then, as viewed from the boat, the rock goes straight up and straight down, as shown in figure 1.2(a). If the same

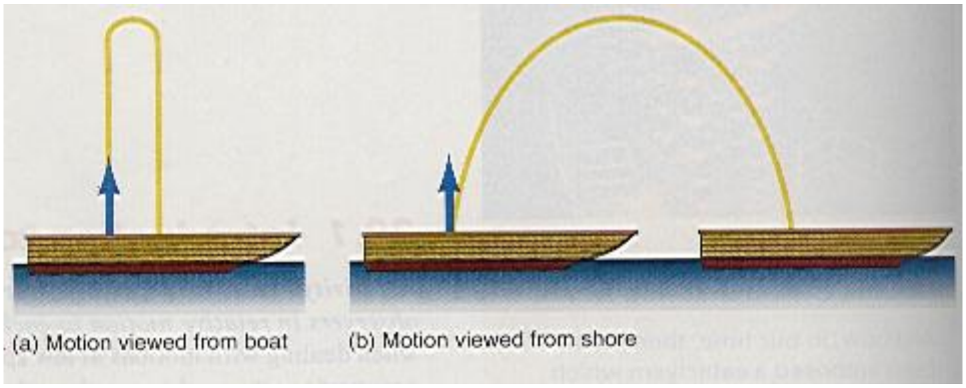


Figure 1.2 Galileo's rebuttal of Aristotle's argument of absolute rest.

projectile motion is observed from the shore, however, the rock is seen to be displaced to the right of the vertical path. The rock comes down to the same place on the boat only because the boat is also moving toward the right. Hence, to the observer on the boat, the rock went straight up and straight down and by Aristotle's reasoning the boat must be at rest. But as the observer on the shore will clearly state, the boat was not at rest but moving with a velocity **v.** Thus, Aristotle's argument is not valid. *The distinction between rest and motion at a constant velocity, is relative to the observer.* The observer on the boat says the boat is at rest while the observer on the shore says the boat is in motion. We then must ask, is there any way to distinguish between a state of rest and a state of motion at constant velocity?

Let us consider Newton's second law of motion as studied in general physics,

$$\mathbf{F} = m\mathbf{a}$$

If the unbalanced external force acting on the body is zero, then the acceleration is also zero. But since $\mathbf{a} = d\mathbf{v}/dt$, this implies that there is no change in velocity of the body, and the velocity is constant. *We are capable of feeling forces and accelerations but we do not feel motion at constant velocity, and rest is the special case of zero constant velocity.* Recall from general physics, concerning the weight of a person in an elevator, the scales read the same numerical value for the weight of the person when the elevator is either at rest or moving at a constant velocity. There is no way for the passenger to say he or she is at rest or moving at a constant velocity unless he or she can somehow look out of the elevator and see motion. When the elevator accelerates upward, on the other hand, the person experiences a greater force pushing upward on him. When the elevator accelerates downward, the person experiences a smaller force on him. Thus, accelerations are easily felt but not constant velocities. Only if the elevator accelerates can the passenger tell that he or she is in motion. While you sit there reading this sentence you are sitting on the earth, which is moving around the sun at about 30 km/s, yet you do not notice this motion.[1] When a person sits in a plane or a train moving at constant velocity, the motion is not sensed unless the person looks out the window. The person senses his or her motion only while the plane or train is accelerating.

Since relative motion depends on the observer, there are many different ways to observe the same motion. For example, figure 1.3(a) shows body 1 at rest while

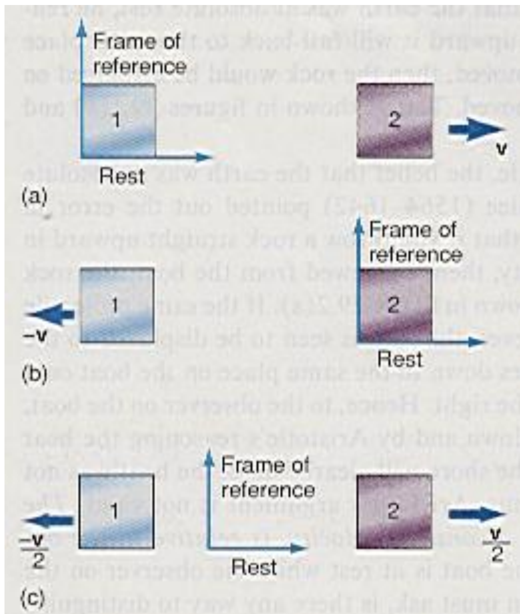


Figure 1.3 Relative motion.

body 2 moves to the right with a velocity **v.** But from the point of view of body 2, he can equally well say that it is he who is at rest and it is body 1 that is moving to the left with the velocity **–v,** figure 1.3(b). Or an arbitrary observer can be placed at rest between bodies 1 and 2, as shown in figure 1.3(c), and she will observe body 2 moving to the right with a velocity **v**/2 and body 1 moving to the left with a velocity of –**v**/2. We can also conceive of the case of body 1 moving to the right with a velocity **v** and body 2 moving to the right with a velocity 2**v,** the relative velocities between the two bodies still being **v** to the right. Obviously an infinite number of such possible cases can be thought out. Therefore, we must conclude that, *if a body in motion at constant velocity is indistinguishable from a body at rest, then there is no reason why a state of rest should be called a state of rest, or a state of motion a state of motion.* Either body can be considered to be at rest while the other body is moving in the opposite direction with the speed v.

To describe the motion, we place a coordinate system at some point, either in the body or outside of it, and call this coordinate system a frame of reference. The motion of any body is then made with respect to this frame of reference. *A frame of reference that is either at rest or moving at a constant velocity is called an inertial*

1[1]Actually the earth's motion around the sun constitutes an accelerated motion. The average centripetal acceleration is $a_c = v^2/r = (33.7 \times 10^3 \text{ m/s})^2/(1.5 \times 10^{11} \text{ m}) = 5.88 \times 10^{-3} \text{ m/s}^2 = 0.0059$ m/s^2. This orbital acceleration is so small compared to the acceleration of gravity, 9.80 m/s^2, that we do not feel it and it can be ignored. Hence, we feel as though we were moving at constant velocity.

frame of reference or an **inertial coordinate system.** *Newton's first law defines the inertial frame of reference.* That is, when **F** = 0, and the body is either at rest or moving uniformly in a straight line, then the body is in an inertial frame. There are an infinite number of inertial frames and Newton's second law, in the form **F** = *m***a,** holds in all these inertial frames.

An example of a noninertial frame is an accelerated frame, and one is shown in figure 1.4. A rock is thrown straight up in a boat that is accelerating to the

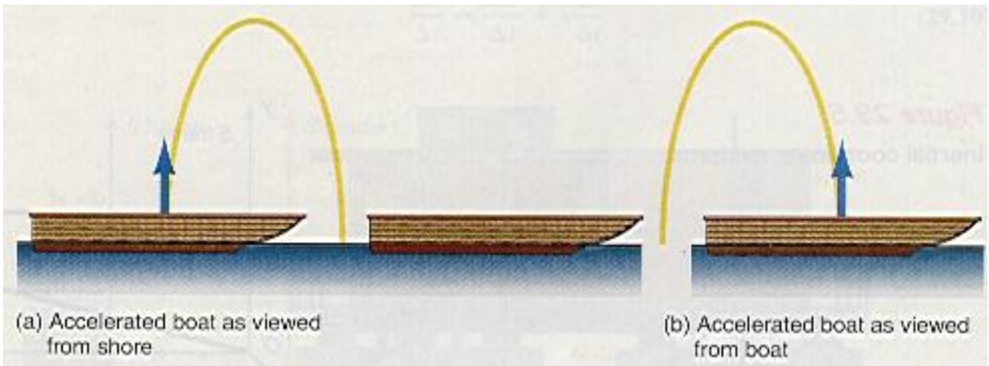


Figure 1.4 A linearly accelerated frame of reference.

right. An observer on the shore sees the projectile motion as in figure 1.4(a). The observed motion of the projectile is the same as in figure 1.2(b), but now the observer on the shore sees the rock fall into the water behind the boat rather than back onto the same point on the boat from which the rock was launched. Because the boat has accelerated while the rock is in the air, the boat has a constantly increasing velocity while the horizontal component of the rock remains a constant. Thus the boat moves out from beneath the rock and when the rock returns to where the boat should be, the boat is no longer there. When the same motion is observed from the boat, the rock does not go straight up and straight down as in figure 1.2(a), but instead the rock appears to move backward toward the end of the boat as though there was a force pushing it backward. The boat observer sees the rock fall into the water behind the boat, figure 1.4(b). In this accelerated reference frame of the boat, there seems to be a force acting on the rock pushing it backward. Hence, Newton's second law, in the form **F** = *m***a,** does not work on this accelerated boat. Instead a fictitious force must be introduced to account for the backward motion of the projectile.

For the moment, we will restrict ourselves to motion as observed from inertial frames of reference, the subject matter of the special or restricted theory of relativity. In chapter 34, we will discuss accelerated frames of reference, the subject matter of general relativity.

1.2 The Galilean Transformations of Classical Physics

The description of any type of motion in classical mechanics starts with an inertial coordinate system S, which is considered to be at rest. Let us consider the occurrence of some "event" that is observed in the S frame of reference, as shown in figure 1.5. The event might be the explosion of a firecracker or the lighting of a

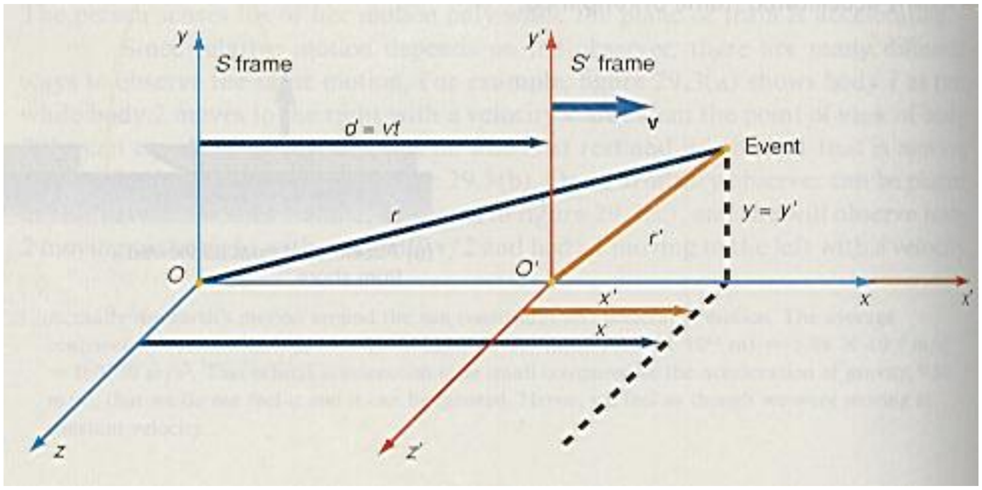


Figure 1.5 Inertial coordinate systems.

match, or just the location of a body at a particular instance of time. For simplicity, we will assume that the event occurs in the x,y plane. The event is located a distance r from the origin O of the S frame. The coordinates of the point in the S frame, are x and y. A second coordinate system S', moving at the constant velocity $\mathbf{v}$ in the positive x-direction, is also introduced. The same event can also be described in terms of this frame of reference. The event is located at a distance r' from the origin O' of the S' frame of reference and has coordinates x' and y', as shown in the figure. We assume that the two coordinate systems had their origins at the same place at the time, $t = 0$. At a later time t, the S' frame will have moved a distance, $d = vt$, along the x-axis. The x-component of the event in the S frame is related to the x'-component of the same event in the S' frame by

$$x = x' + vt \tag{1.1}$$

which can be easily seen in figure 1.5, and the y- and y'-components are seen to be

$$y = y' \tag{1.2}$$

Notice that because of the initial assumption, z and z' are also equal, that is

$$z = z' \tag{1.3}$$

It is also assumed, but usually never stated, that the time is the same in both frames of reference, that is,

$$t = t' \tag{1.4}$$

These equations, that describe the event from either inertial coordinate system, are called the ***Galilean transformations*** *of classical mechanics and they are summarized as*

$$x = x' + vt \tag{1.1}$$
$$y = y' \tag{1.2}$$
$$z = z' \tag{1.3}$$
$$t = t' \tag{1.4}$$

The inverse transformations from the S frame to the S' frame are

$$x' = x - vt \tag{1.5}$$
$$y' = y \tag{1.6}$$
$$z' = z \tag{1.7}$$
$$t' = t \tag{1.8}$$

Example 1.1

The Galilean transformation of distances. A student is sitting on a train 10.0 m from the rear of the car. The train is moving to the right at a speed of 4.00 m/s. If the rear of the car passes the end of the platform at $t = 0$, how far away from the platform is the student at 5.00 s?

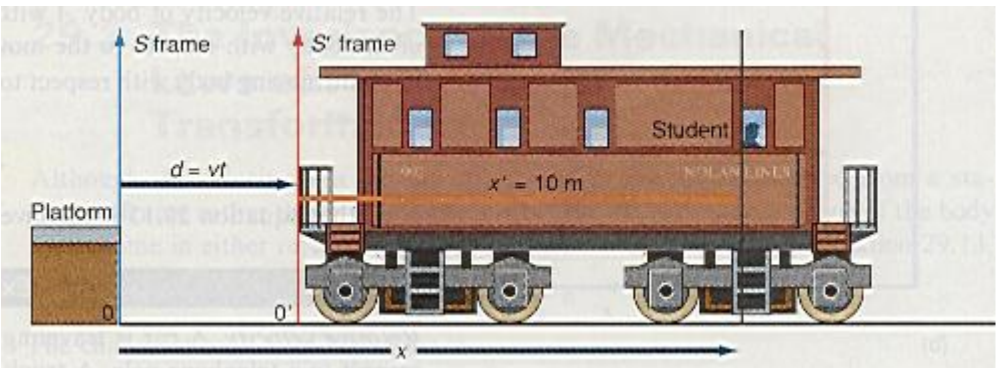


Figure 1.6 An example of the Galilean transformation.

Solution

The picture of the student, the train, and the platform is shown in figure 1.6. The platform represents the stationary S frame, whereas the train represents the moving S' frame. The location of the student, as observed from the platform, found from equation 1.1, is

$$x = x' + vt$$
$$= 10.0 \text{ m} + (4.00 \text{ m/s})(5.00 \text{ s})$$
$$= 30 \text{ m}$$

Go to Interactive Example

The speed of an object in either frame can be easily found by differentiating the Galilean transformation equations with respect to t. That is, for the x-component of the transformation we have

$$x = x' + vt$$

Upon differentiating

$$\frac{dx}{dt} = \frac{dx'}{dt} + v\frac{dt}{dt} \tag{1.9}$$

But $dx/dt = v_x$, the x-component of the velocity of the body in the stationary frame S, and $dx'/dt = v'_x$, the x-component of the velocity in the moving frame S'. Thus equation 1.9 becomes

$$v_x = v'_x + v \tag{1.10}$$

Equation 1.10 is a statement of the Galilean addition of velocities.

Example 1.2

The Galilean transformation of velocities. The student on the train of example 1.1, gets up and starts to walk. What is the student's speed relative to the platform if (a) the student walks toward the front of the train at a speed of 2.00 m/s and (b) the student walks toward the back of the train at a speed of 2.00 m/s?

Solution

a. The speed of the student relative to the stationary platform, found from equation 1.10, is

$$v_x = v'_x + v = 2.00 \text{ m/s} + 4.00 \text{ m/s}$$
$$= 6.00 \text{ m/s}$$

b. If the student walks toward the back of the train $\Delta x' = x'_2 - x'_1$ is negative because x'_1 is greater than x'_2, and hence, v'_x is a negative quantity. Therefore,

$$v_x = v'_x + v$$
$$= -2.00 \text{ m/s} + 4.00 \text{ m/s}$$
$$= 2.00 \text{ m/s}$$

Go to Interactive Example

If there is more than one body in motion with respect to the stationary frame, the relative velocity between the two bodies is found by placing the S' frame on one of the bodies in motion. That is, if body *A* is moving with a velocity $\mathbf{v}_{AS}$ with respect to the stationary frame *S,* and body *B* is moving with a velocity $\mathbf{v}_{BS}$, also with respect to the stationary frame *S,* the velocity of *A* as observed from *B,* $\mathbf{v}_{AB}$, is simply

$$\mathbf{v}_{AB} = \mathbf{v}_{AS} - \mathbf{v}_{BS} \tag{1.11}$$

as seen in figure 1.7(a).

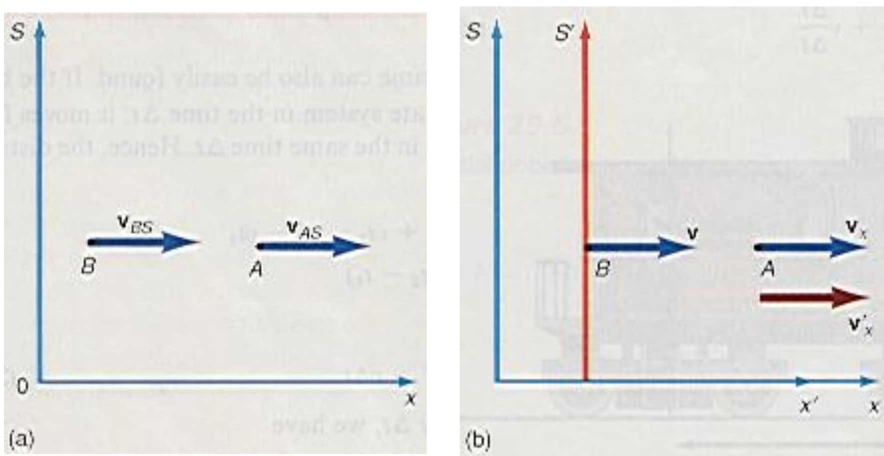


Figure 1.7 Relative velocities.

If we place the moving frame of reference *S'* on body *B,* as in figure 1.7(b), then $\mathbf{v}_{BS} = \mathbf{v}$, the velocity of the *S'* frame. The velocity of the body *A* with respect to *S,* $\mathbf{v}_{AS}$, is now set equal to $\mathbf{v}_x$, the velocity of the body with respect to the *S* frame. The relative velocity of body *A* with respect to body *B,* $\mathbf{v}_{AB}$, is now $\mathbf{v}'_x$, the velocity of the body with respect to the moving frame of reference *S'*. Hence the velocity $\mathbf{v}'_x$ of the moving body with respect to the moving frame is determined from equation 1.11 as

$$\mathbf{v}'_x = \mathbf{v}_x - \mathbf{v} \tag{1.12}$$

Note that equation 1.12 is the inverse of equation 1.10.

Example 1.3

Relative velocity. A car is traveling at a velocity of 95.0 km/hr to the right, with respect to a telephone pole. A truck, which is behind the car, is also moving to the right at 65.0 km/hr with respect to the same telephone pole. Find the relative velocity of the car with respect to the truck.

Solution

We represent the telephone pole as the stationary frame of reference S, while we place the moving frame of reference S' on the truck that is moving at a speed v = 65.0 km/hr. The auto is moving at the speed v = 95.0 km/hr with respect to S. The velocity of the auto with respect to the truck (or S' frame) is v'_x and is found from equation 1.12 as

$$\begin{aligned} v'_x &= v_x - v \\ &= 95.0 \text{ km/hr} - 65.0 \text{ km/hr} \\ &= 30.0 \text{ km/hr} \end{aligned}$$

The relative velocity is +30.0 km/hr. This means that the auto is pulling away or separating from the truck at the rate of 30.0 km/hr. If the auto were moving toward the S observer instead of away, then the auto's velocity with respect to S' would have been

$$\begin{aligned} v'_x = v_x - v &= -95.0 \text{ km/hr} - 65.0 \text{ km/hr} \\ &= -160.0 \text{ km/hr} \end{aligned}$$

That is, the truck would then observe the auto approaching at a closing speed of –160 km/hr. Note that when the relative velocity v'_x is positive the two moving objects are separating, whereas when v'_x is negative the two objects are closing or coming toward each other.

Go to Interactive Example

To complete the velocity transformation equations, we use the fact that $y = y'$ and $z = z'$, thereby giving us

$$v'_y = \frac{dy'}{dt} = \frac{dy}{dt} = v_y \tag{1.13}$$

and

$$v'_z = \frac{dz'}{dt} = \frac{dz}{dt} = v_z \tag{1.14}$$

The Galilean transformations of velocities can be summarized as:

$$v_x = v'_x + v \tag{1.10}$$
$$v'_x = v_x - v \tag{1.12}$$
$$v'_y = v_y \tag{1.13}$$
$$v'_z = v_z \tag{1.14}$$

1.3 The Invariance of the Mechanical Laws of Physics under a Galilean Transformation

Although the velocity of a moving object is different when observed from a stationary frame rather than a moving frame of reference, the acceleration of the body is the same in either reference frame. To see this, let us start with equation 1.12,

$$v'_x = v_x - v$$

The change in each term with time is

$$\frac{dv'_x}{dt} = \frac{dv_x}{dt} - \frac{dv}{dt} \tag{1.15}$$

But v is the speed of the moving frame, which is a constant and does not change with time. Hence, $dv/dt = 0$. The term $dv'_x/dt = a'_x$ is the acceleration of the body with respect to the moving frame, whereas $dv_x/dt = a_x$ is the acceleration of the body with respect to the stationary frame. Therefore, equation 1.15 becomes

$$a'_x = a_x \tag{1.16}$$

Equation 1.16 says that the acceleration of a moving body is invariant under a Galilean transformation. *The word invariant when applied to a physical quantity means that the quantity remains a constant. We say that the acceleration is an* ***invariant quantity.*** *This means that either the moving or stationary observer would measure the same numerical value for the acceleration of the body.*

If we multiply both sides of equation 1.16 by m, we get

$$ma'_x = ma_x \tag{1.17}$$

But the product of the mass and the acceleration is equal to the force F, by Newton's second law. Hence,

$$F' = F \tag{1.18}$$

Thus, Newton's second law is also invariant to a Galilean transformation and applies to all inertial observers.

The laws of conservation of momentum and conservation of energy are also invariant under a Galilean transformation. We can see this for the case of the perfectly elastic collision illustrated in figure 1.8. We can write the law of conservation of momentum for the collision, as observed in the S frame, as

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{V}_1 + m_2\mathbf{V}_2 \tag{1.19}$$

where v_1 is the velocity of ball 1 before the collision, v_2 is the velocity of ball 2 before the collision, V_1 is the velocity of ball 1 after the collision, and V_2 is the velocity of ball 2 after the collision. But the relation between the velocity in the S and S' frames, found from equation 1.11 and figure 1.8, is

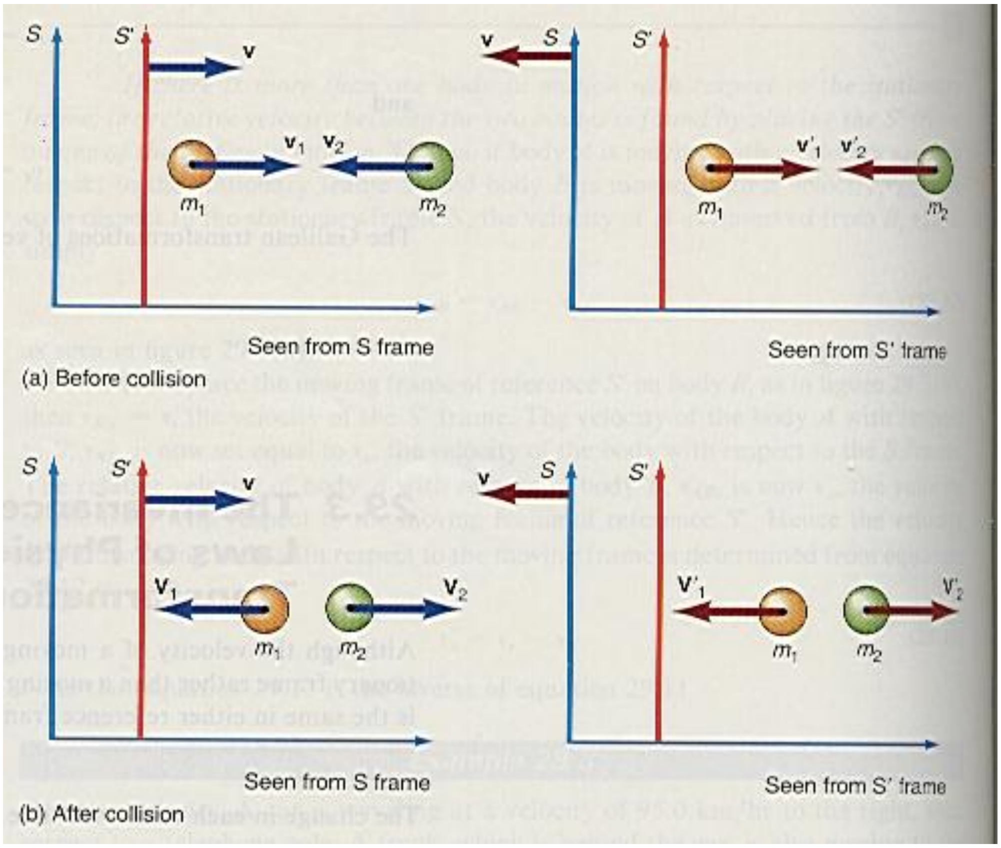


Figure 1.8 A perfectly elastic collision as seen from two inertial frames.

$$\left.\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_1' + \mathbf{v} \\ \mathbf{v}_2 &= \mathbf{v}_2' + \mathbf{v} \\ \mathbf{V}_1 &= \mathbf{V}_1' + \mathbf{v} \\ \mathbf{V}_2 &= \mathbf{V}_2' + \mathbf{v} \end{aligned}\right\} \tag{1.20}$$

Substituting equations 1.20 into equation 1.19 for the law of conservation of momentum yields

$$m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2 = m_1\mathbf{V}'_1 + m_2\mathbf{V}'_2 \tag{1.21}$$

Equation 1.21 is the law of conservation of momentum as observed from the moving S' frame. Note that it is of the same form as the law of conservation of momentum as observed from the S or stationary frame of reference. Thus, *the law of conservation of momentum is invariant to a Galilean transformation.*

The law of conservation of energy for the perfectly elastic collision of figure 1.8 as viewed from the S frame is

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 \tag{1.22}$$

By replacing the velocities in equation 1.22 by their Galilean counterparts, equation 1.20, and after much algebra we find that

$$\frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \tag{1.23}$$

Equation 1.23 is the law of conservation of energy as observed by an observer in the moving S' frame of reference. Note again that the form of the equation is the same as in the stationary frame, and hence, *the law of conservation of energy is invariant to a Galilean transformation.* If we continued in this manner we would prove that all the laws of mechanics are invariant to a Galilean transformation.

1.4 Electromagnetism and the Ether

We have just seen that the laws of mechanics are invariant to a Galilean transformation. Are the laws of electromagnetism also invariant?

Consider a spherical electromagnetic wave propagating with a speed c with respect to a stationary frame of reference, as shown in figure 1.9. The speed of this

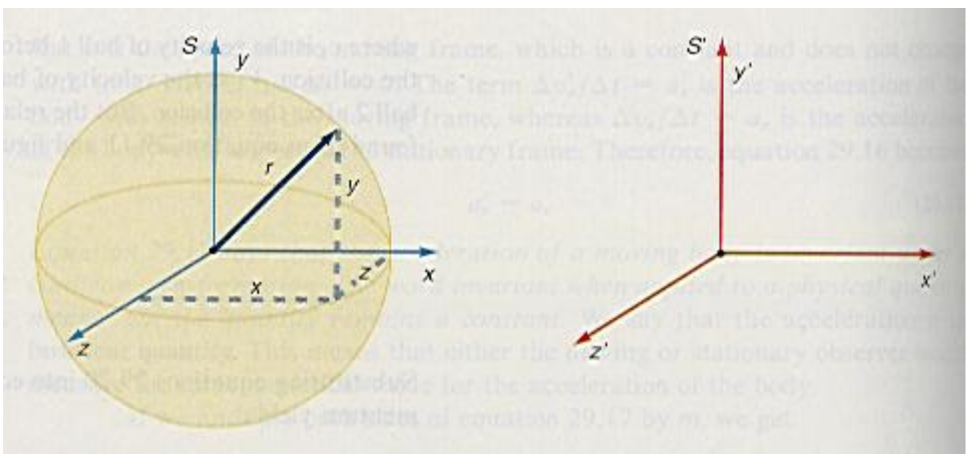


Figure 1.9 A spherical electromagnetic wave.

electromagnetic wave is

$$c = \frac{r}{t}$$

where r is the distance from the source of the wave to the spherical wave front. We can rewrite this as

$$r = ct$$

or

$$r^2 = c^2t^2$$

or

$$r^2 - c^2t^2 = 0 \tag{1.24}$$

The radius r of the spherical wave is

$$r^2 = x^2 + y^2 + z^2$$

Substituting this into equation 1.24, gives

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{1.25}$$

for the light wave as observed in the S frame of reference. Let us now assume that another observer, moving at the speed v in a moving frame of reference S' also observes this same light wave. The S' observer observes the coordinates x' and t', which are related to the x and t coordinates by the Galilean transformation equations as

$$x = x' + vt \tag{1.1}$$
$$y = y' \tag{1.2}$$
$$z = z' \tag{1.3}$$
$$t = t' \tag{1.4}$$

Substituting these Galilean transformations into equation 1.25 gives

$$(x' + vt)^2 + y'^2 + z'^2 - c^2t'^2 = 0$$
$$x'^2 + 2x'vt + v^2t^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

or

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = -2x'vt - v^2t^2 \tag{1.26}$$

Notice that the form of the equation is not invariant to a Galilean transformation. That is, equation 1.26, the velocity of the light wave as observed in the S' frame, has a different form than equation 1.25, the velocity of light in the S frame. Something is very wrong either with the equations of electromagnetism or with the Galilean transformations. Einstein was so filled with the beauty of the unifying effects of Maxwell's equations of electromagnetism that he felt that there must be something wrong with the Galilean transformation and hence, a new transformation law was required.

A further difficulty associated with the electromagnetic waves of Maxwell was the medium in which these waves propagated. Recall from your general physics course, that a wave is a disturbance that propagates through a medium. When a rock, the disturbance, is dropped into a pond, a wave propagates through the water, the medium. Associated with a transverse wave on a string is the motion of the particles of the string executing simple harmonic motion perpendicular to the direction of the wave propagation. In this case, the medium is the particles of the string. A sound wave in air is a disturbance propagated through the medium air. In fact, when we say that a sound wave propagates through the air with a velocity of 330 m/s at 0 ^{0}C, we mean that the wave is moving at 330 m/s with respect to the air.

A sound wave in water propagates through the water while a sound wave in a solid propagates through the solid. The one thing that all of these waves have in common is that they are all propagated through some medium. Classical physicists then naturally asked, "Through what medium does light propagate?" According to everything that was known in the field of physics in the nineteenth century, a wave must propagate through some medium. Therefore, it was reasonable to expect that a light wave, like any other wave, must propagate through some medium. This medium was called the luminiferous ether or just **ether** for short. It was assumed that this ether filled all of space, the inside of all material bodies, and was responsible for the transmission of all electromagnetic vibrations. Maxwell assumed that his electromagnetic waves propagated through this ether at the speed $c = 3 \times 10^8$ m/s.

An additional reason for the assumption of the existence of the ether was the phenomena of interference and diffraction of light that implied that light must be a wave. If light is an electromagnetic wave, then it is waving through the medium called ether.

There are, however, two disturbing characteristics of this ether. First, the ether had to have some very strange properties. The ether had to be very strong or rigid in order to support the extremely large speed of light. Recall from your general physics course that the speed of sound at 0 ^{0}C is 330 m/s in air, 1520 m/s in water, and 3420 m/s in iron. Thus, the more rigid the medium the higher the velocity of the wave. Similarly, for a transverse wave on a taut string the speed of propagation is

$$v = \sqrt{\frac{T}{m/l}}$$

where T is the tension in the string. The greater the value of T, the greater the value of the speed of propagation. Greater tension in the string implies a more rigid string. Although a light wave is neither a sound wave nor a wave on a string, it is reasonable to assume that the ether, being a medium for propagation of an electromagnetic wave, should also be quite rigid in order to support the enormous speed of 3×10^8 m/s. Yet the earth moves through this rigid medium at an orbital speed of 3×10^4 m/s and its motion is not impeded one iota by this rigid medium. This is very strange indeed.

The second disturbing characteristic of this ether hypothesis is that if electromagnetic waves always move at a speed c with respect to the ether, then maybe the ether constitutes an absolute frame of reference that we have not been able to find up to now. *Newton postulated an absolute space and an absolute time in his Principia: "Absolute space, in its own nature without regard to anything external remains always similar and immovable." And, "Absolute, true, and mathematical time, of itself and from its own nature flows equally without regard to anything external."* Could the ether be the framework of absolute space? In order to settle these apparent inconsistencies, it became necessary to detect this medium, called the ether, and thus verify its very existence. Maxwell suggested a crucial experiment to detect this ether. The experiment was performed by A. A. Michelson and E. E. Morley and is described in section 1.5.

1.5 The Michelson-Morley Experiment

If there is a medium called the ether that pervades all of space then the earth must be moving through this ether as it moves in its orbital motion about the sun. From the point of view of an observer on the earth the ether must flow past the earth, that is, it must appear that the earth is afloat in an ether current. The ether current concept allows us to consider an analogy of a boat in a river current.

Consider a boat in a river, L meters wide, where the speed of the river current is some unknown quantity v, as shown in figure 1.10. The boat is capable

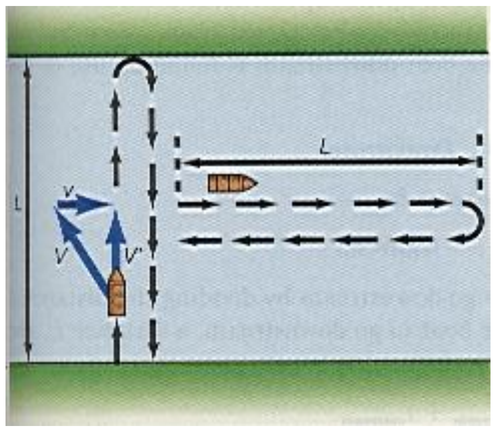


Figure 1.10 Current flowing in a river.

of moving at a speed V with respect to the water. The captain would like to measure the river current v, using only his stopwatch and the speed of his boat with respect to the water. After some thought the captain proceeds as follows. He can measure the time it takes for his boat to go straight across the river and return. But if he heads straight across the river, the current pushes the boat downstream. Therefore, he heads the boat upstream at an angle such that one component of the boat's velocity with respect to the water is equal and opposite to the velocity of the current downstream. Hence, the boat moves directly across the river at a velocity **V'**, as shown in the figure. The speed V' can be found from the application of the Pythagorean theorem to the velocity triangle of figure 1.10, namely

$$V^2 = V'^2 + v^2$$

Solving for V', we get

$$V' = \sqrt{V^2 - v^2}$$

Factoring out a V, we obtain, for the speed of the boat across the river,

$$V' = V\sqrt{1 - v^2/V^2} \tag{1.27}$$

We find the time to cross the river by dividing the distance traveled by the boat by the boat's speed, that is,

$$t_{\text{across}} = \frac{L}{V'}$$

The time to return is the same, that is,

$$t_{\text{return}} = \frac{L}{V'}$$

Hence, the total time to cross the river and return is

$$t_1 = t_{\text{across}} + t_{\text{return}} = \frac{L}{V'} + \frac{L}{V'} = \frac{2L}{V'}$$

Substituting V' from equation 1.27, the time becomes

$$t_1 = \frac{2L}{V\sqrt{1 - v^2/V^2}}$$

Hence, the time for the boat to cross the river and return is

$$t_1 = \frac{2L/V}{\sqrt{1 - v^2/V^2}} \tag{1.28}$$

The captain now tries another motion. He takes the boat out to the middle of the river and starts the boat downstream at the same speed V with respect to the water. After traveling a distance L downstream, the captain turns the boat around and travels the same distance L upstream to where he started from, as we can see in figure 1.10. The actual velocity of the boat downstream is found by use of the Galilean transformation as

$$V' = V + v \quad \text{Downstream}$$

while the actual velocity of the boat upstream is

$$V' = V - v \qquad \text{Upstream}$$

We find the time for the boat to go downstream by dividing the distance L by the velocity V'. Thus the time for the boat to go downstream, a distance L, and to return is

$$t_2 = t_{\text{downstream}} + t_{\text{upstream}}$$

$$= \frac{L}{V + v} + \frac{L}{V - v}$$

Finding a common denominator and simplifying,

$$t_2 = \frac{L(V - v) + L(V + v)}{(V + v)(V - v)}$$

$$= \frac{LV - Lv + LV + Lv}{V^2 + vV - vV - v^2}$$

$$= \frac{2LV}{V^2 - v^2}$$

$$= \frac{2LV/V^2}{V^2/V^2 - v^2/V^2}$$

$$t_2 = \frac{2L/V}{1 - v^2/V^2} \tag{1.29}$$

Hence, t_2 in equation 1.29 is the time for the boat to go downstream and return. Note from equations 1.28 and 1.29 that the two travel times are not equal.

The ratio of t_1, the time for the boat to cross the river and return, to t_2, the time for the boat to go downstream and return, found from equations 1.28 and 1.29, is

$$\frac{t_1}{t_2} = \frac{(2L/V)/\sqrt{1 - v^2/V^2}}{(2L/V)/\left(1 - v^2/V^2\right)}$$

$$= \frac{\left(1 - v^2/V^2\right)}{\sqrt{1 - v^2/V^2}}$$

$$\frac{t_1}{t_2} = \sqrt{1 - v^2/V^2} \tag{1.30}$$

Equation 1.30 says that if the speed v of the river current is known, then a relation between the times for the two different paths can be determined. On the other hand, if t_1 and t_2 are measured and the speed of the boat with respect to the water V is known, then the speed of the river current v can be determined. Thus, squaring equation 1.30,

$$\frac{t_1^2}{t_2^2} = 1 - \frac{v^2}{V^2}$$

$$\frac{v^2}{V^2} = 1 - \frac{t_1^2}{t_2^2}$$

or

$$v = V\sqrt{1 - \frac{t_1^2}{t_2^2}} \tag{1.31}$$

Thus, by knowing the times for the boat to travel the two paths the speed of the river current v can be determined.

Using the above analogy can help us to understand the experiment performed by Michelson and Morley to detect the ether current. The equipment used to measure the ether current was the Michelson interferometer and is sketched in figure 1.11. The interferometer sits in a laboratory on the earth. Because the earth moves through the ether, the observer in the laboratory sees an ether current moving past him with a speed of approximately $v = 3.00 \times 10^4$ m/s,

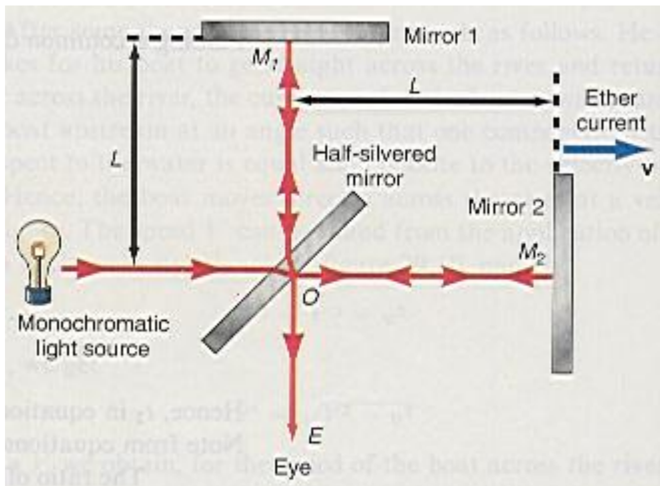


Figure 1.11 The Michelson-Morley experiment.

the orbital velocity of the earth about the sun. The motion of the light throughout the interferometer is the same as the motion of the boat in the river current. Light from the extended source is split by the half-silvered mirror. Half the light follows the path OM_1OE, which is perpendicular to the ether current. The rest follows the path OM_2OE, which is first in the direction of the ether current until it is reflected from mirror M_2, and is then in the direction that is opposite to the ether current. *The time for the light to cross the ether current is found from equation 1.28, but with V the speed of the boat replaced by c, the speed of light.* Thus,

$$t_1 = \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

The time for the light to go downstream and upstream in the ether current is found from equation 1.29 but with V replaced by c. Thus,

$$t_2 = \frac{2L/c}{1 - v^2/c^2}$$

The time difference between the two optical paths because of the ether current is

$$\Delta t = t_2 - t_1$$

$$\Delta t = \frac{2L/c}{1 - v^2/c^2} - \frac{2L/c}{\sqrt{1 - v^2/c^2}} \tag{1.32}$$

To simplify this equation, we use the *binomial theorem*. That is,

$$(1 - x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots \tag{1.33}$$

This is a valid series expansion for $(1 - x)^n$ as long as x is less than 1. In this particular case,

$$x = \frac{v^2}{c^2} = \frac{(3.00 \times 10^4 \text{ m/s})^2}{(3.00 \times 10^8 \text{ m/s})^2} = 10^{-8}$$

which is much less than 1. In fact, since $x = 10^{-8}$, which is very small, it is possible to simplify the binomial theorem to

$$(1 - x)^n = 1 - nx \tag{1.34}$$

That is, since $x = 10^{-8}$, $x^2 = 10^{-16}$, and $x^3 = 10^{-24}$, the terms in x^2 and x^3 are negligible when compared to the value of x, and can be set equal to zero. Therefore, we can write the denominator of the first term in equation 1.32 as

$$\frac{1}{1 - v^2/c^2} = \left(1 - \frac{v^2}{c^2}\right)^{-1} = 1 - (-1)\frac{v^2}{c^2} = 1 + \frac{v^2}{c^2} \tag{1.35}$$

The denominator of the second term can be expressed as

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 - \left(-\frac{1}{2}\right)\frac{v^2}{c^2}$$

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1v^2}{2c^2} \tag{1.36}$$

Substituting equations 1.35 and 1.36 into equation 1.32, yields

$$\begin{aligned} \Delta t &= \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2}\right) \\ &= \frac{2L}{c}\left(1 + \frac{v^2}{c^2} - 1 - \frac{1v^2}{2c^2}\right) \\ &= \frac{2L}{c}\left(\frac{1v^2}{2c^2}\right) \end{aligned}$$

The path difference d between rays OM_1OE and OM_2OE, corresponding to this time difference Δt, is

$$d = c\Delta t = c\left[\frac{2L}{c}\left(\frac{1v^2}{2c^2}\right)\right]$$

or

$$d = \frac{Lv^2}{c^2} \tag{1.37}$$

Equation 1.37 gives the path difference between the two light rays and would cause the rays of light to be out of phase with each other and should cause an interference fringe. However, as explained in your optics course, the mirrors M_1 and M_2 of the Michelson interferometer are not quite perpendicular to each other and we always get interference fringes. However, if the interferometer is rotated through 90^0, then the optical paths are interchanged. That is, the path that originally required a time t_1 for the light to pass through, now requires a time t_2 and vice versa. The new time difference between the paths, analogous to equation 1.32, becomes

$$\Delta t' = \frac{2L/c}{\sqrt{1 - v^2/c^2}} - \frac{2L/c}{1 - v^2/c^2}$$

Using the binomial theorem again, we get

$$\begin{aligned}\Delta t' &= \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2}\right) - \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right)\\ &= \frac{2L}{c}\left(1 + \frac{1v^2}{2c^2} - 1 - \frac{v^2}{c^2}\right)\\ &= \frac{2L}{c}\left(-\frac{1v^2}{2c^2}\right)\\ &= -\frac{Lv^2}{cc^2}\end{aligned}$$

The difference in path corresponding to this time difference is

$$d' = c\Delta t' = c\left(-\frac{Lv^2}{cc^2}\right)$$

or

$$d' = -\frac{Lv^2}{c^2}$$

By rotating the interferometer, the optical path has changed by

$$\Delta d = d - d' = \frac{Lv^2}{c^2} - \left(-\frac{Lv^2}{c^2}\right)$$

$$\Delta d = \frac{2Lv^2}{c^2} \tag{1.38}$$

This change in the optical paths corresponds to a shifting of the interference fringes. That is,

$$\Delta d = \Delta n\, \lambda$$

or

$$\Delta n = \frac{\Delta d}{\lambda} \tag{1.39}$$

Using equations 1.38 for Δd, *the number of fringes, Δn, that should move across the screen when the interferometer is rotated is*

$$\Delta n = \frac{2Lv^2}{\lambda\, c^2} \tag{1.40}$$

In the actual experimental set-up, the light path L was increased to 10.0 m by multiple reflections. The wavelength of light used was 500.0 nm. The ether current was assumed to be 3.00×10^4 m/s, the orbital speed of the earth around the sun. When all these values are placed into equation 1.40, the expected fringe shift is

$$\Delta n = \frac{2(10.0 \text{ m})(3.00 \times 10^4 \text{ m/s})^2}{(5.000 \times 10^{-7} \text{ m})(3.00 \times 10^8 \text{ m/s})^2}$$
$$= 0.400 \text{ fringes}$$

That is, if there is an ether that pervades all space, the earth must be moving through it. ***This ether current should cause a fringe shift of 0.400 fringes in the rotated interferometer, however, no fringe shift whatsoever was found.*** It should be noted that the interferometer was capable of reading a shift much smaller than the 0.400 fringe expected.

On the rare possibility that the earth was moving at the same speed as the ether, the experiment was repeated six months later when the motion of the earth was in the opposite direction. Again, no fringe shift was observed. ***The ether cannot be detected. But if it cannot be detected there is no reason to even assume that it exists.*** *Hence, the* ***Michelson-Morley experiment's*** *null result implies that the all pervading medium called the ether simply does not exist.* Therefore light, and all electromagnetic waves, are capable of propagating without the use of any medium. If there is no ether then the speed of the ether wind v is equal to zero. The null result of the experiment follows directly from equation 1.40 with $v = 0$.

The negative result also suggested a new physical principle. Even if there is no ether, when the light moves along path OM_2 the Galilean transformation equations with respect to the "fixed stars" still imply that the velocity of light along OM_2 should be $c + v$, where v is the earth's orbital velocity, with respect to the fixed stars, and c is the velocity of light with respect to the source on the interferometer. Similarly, it should be $c - v$ along path M_2O. But the negative result of the experiment requires the light to move at the same speed c whether the light was moving with the earth or against it. Hence, ***the negative result implies that the speed of light in free space is the same everywhere regardless of the motion of the source or the observer.*** *This also implies that there is something wrong with the Galilean transformation, which gives us the $c + v$ and $c - v$ velocities. Thus, it would appear that a new transformation equation other than the Galilean transformation is necessary.*

1.6 The Postulates of the Special Theory of Relativity

In 1905, Albert Einstein (1879-1955) formulated his **Special or Restricted Theory of Relativity** in terms of two postulates.

> *Postulate 1: The laws of physics have the same form in all frames of reference moving at a constant velocity with respect to one another.* This first postulate is sometimes also stated in the more succinct form: The laws of physics are invariant to a transformation between all inertial frames.
> *Postulate 2: The speed of light in free space has the same value for all observers, regardless of their state of motion.*

Postulate 1 is, in a sense, a consequence of the fact that all inertial frames are equivalent. If the laws of physics were different in different frames of reference, then we could tell from the form of the equation used which frame we were in. In particular, we could tell whether we were at rest or moving. But the difference between rest and motion at a constant velocity cannot be detected. Therefore, the laws of physics must be the same in all inertial frames.

Postulate 2 says that the velocity of light is always the same independent of the velocity of the source or of the observer. This can be taken as an experimental fact deduced from the Michelson-Morley experiment. However, Einstein, when asked years later if he had been aware of the results of the Michelson-Morley experiment, replied that he was not sure if he had been. Einstein came on the second postulate from a different viewpoint. According to his first postulate, the laws of physics must be the same for all inertial observers. If the velocity of light is different for different observers, then the observer could tell whether he was at rest or in motion at some constant velocity, simply by determining the velocity of light in his frame of reference. If the observed velocity of light c' were equal to c then the observer would be in the frame of reference that is at rest. If the observed velocity of light were $c' = c - v$, then the observer was in a frame of reference that was receding from the rest frame. Finally, if the observed velocity $c' = c + v$, then the observer would be in a frame of reference that was approaching the rest frame. Obviously these various values of c' would be a violation of the first postulate, since we could now define an absolute rest frame ($c' = c$), which would be different than all the other inertial frames.

The second postulate has revolutionary consequences. Recall that a velocity is equal to a distance in space divided by an interval of time. *In order for the velocity of light to remain a constant independent of the motion of the source or observer, space and time itself must change.* This is a revolutionary concept, indeed, because as already pointed out, Newton had assumed that space and time were absolute. A length of 1 m was considered to be a length of 1 m anywhere, and a time interval of 1 hr was considered to be a time interval of 1 hr anywhere. However, *if space and time change, then these concepts of absolute space and absolute time can no longer be part of the picture of the physical universe.*

The negative results of the Michelson-Morley experiment can also be explained by the second postulate. The velocity of light must always be *c,* never the $c + v$, $c - v$, or $\sqrt{c^2 - v^2}$ that were used in the original derivation. Thus, there would be no difference in time for either optical path of the interferometer and no fringe shift.

The Galilean equations for the transformation of velocity, which gave us the velocities of light as $c' = c + v$ and $c' = c - v$, must be replaced by some new transformation that always gives the velocity of light as c regardless of the velocity of the source or the observer. In section 1.7 we will derive such a transformation.

1.7 The Lorentz Transformation

Because the Galilean transformations violate the postulates of relativity, we must derive a new set of equations that relate the position and velocity of an object in one inertial frame to its position and velocity in another inertial frame. And we must derive the new transformation equations directly from the postulates of special relativity.

Since the Galilean transformations are correct when dealing with the motion of a body at low speeds, the new equations should reduce to the Galilean equations at low speeds. Therefore, the new transformation should have the form

$$x' = k(x - vt) \tag{1.41}$$

where x is the position of the body in the "rest" frame, t is the time of its observation, x' is the position of the body in the moving frame of reference, and finally k is some function or constant to be determined. For the classical case of low speeds, k should reduce to the value 1, and the new transformation equation would then reduce to the Galilean transformation, equation 1.5. This equation says that if the position x and velocity v of a body are measured in the stationary frame, then its position x' in the moving frame is determined by equation 1.41. Using the first postulate of relativity, this equation must have the same form in the frame of reference at rest. Therefore,

$$x = k(x' + vt') \tag{1.42}$$

where *x'* is the position of the body in the moving frame at the time *t'*. The sign of *v* has been changed to a positive quantity because, as shown in figure 1.3, a frame 2 moving to the right with a velocity *v* as observed from a frame 1 at rest, is equivalent to frame 2 at rest with frame 1 moving to the left with a velocity $-v$. This equation says that if the position *x'* and time *t'* of a body are measured in a moving frame, then its position *x* in the stationary frame is determined by equation 1.42. The position of the *y*- and *z*-coordinates are still the same, namely,

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned} \tag{1.43}$$

The time of the observation of the event in the moving frame is denoted by *t'*. We deliberately depart from our common experiences by arranging for the possibility of a different time *t'* for the event in the moving frame compared to the time *t* for the same event in the stationary frame. In fact, *t'* can be determined by substituting equation 1.41 into equation 1.42. That is,

$$\begin{aligned} x = k(x' + vt') &= k[k(x - vt) + vt'] \\ &= k^2x - k^2vt + kvt' \\ kvt' &= x - k^2x + k^2vt \end{aligned}$$

and

$$t' = kt + \left(\frac{1-k^2}{kv}\right)x \tag{1.44}$$

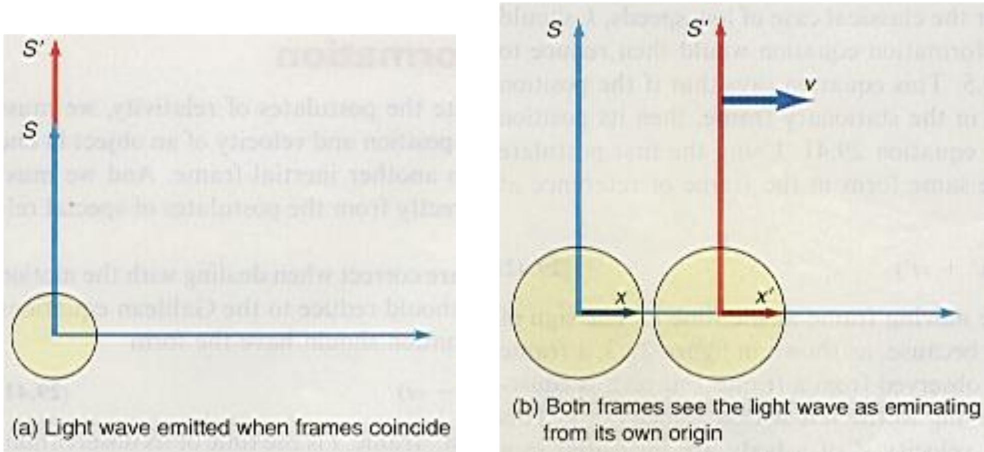
Thus, according to the results of the first postulate of relativity, the time *t'* in the moving coordinate system is not equal to the time *t* in the stationary coordinate system. The exact relation between these times is still unknown, however, because we still have to determine the value of *k*.

To determine *k,* we use the second postulate of relativity. Imagine a light wave emanating from a source that is located at the origin of the *S* and *S'* frame of reference, which momentarily coincide for $t = 0$ and $t' = 0$, figure 1.12. By the second postulate both the stationary and moving observer must observe the same velocity *c* of the light wave. The distance the wave moves in the *x*-direction in the *S* frame is

$$x = ct \tag{1.45}$$

whereas the distance the same wave moves in the *x'*-direction in the *S'* frame is

$$x' = ct' \tag{1.46}$$



(a) Light wave emitted when frames coincide
(b) Botn frames see the light wave as eminating from its own origin

Figure 1.12 The same light wave observed from two inertial frames.

Substituting for x' from equation 1.41, and for t' from equation 1.44, into equation 1.46, yields

$$k(x - vt) = c\left[kt + \left(\frac{1-k^2}{kv}\right)x\right]$$

Performing the following algebraic steps, we solve for x

$$kx - kvt = ckt + \frac{c(1-k^2)}{kv}x$$

$$kx - \frac{c(1-k^2)}{kv}x = ckt + kvt$$

$$x\left[k - \frac{c(1-k^2)}{kv}\right] = ct\left(k + \frac{kv}{c}\right)$$

$$x = ct\left[\frac{k + kv/c}{k - c\left[(1-k^2)/kv\right]}\right] \tag{1.47}$$

But as already seen in equation 1.45, $x = ct$. Therefore, the term in braces in equation 1.47 must be equal to 1. Thus,

$$\frac{k + kv/c}{k - [c(1-k^2)/kv]} = 1$$

$$\frac{k(1 + v/c)}{k\{1 - [c(1-k^2)/k^2v]\}} = 1$$

$$1 + \frac{v}{c} = 1 - \frac{c}{v}\left(\frac{1}{k^2} - 1\right) = 1 - \frac{c}{vk^2} + \frac{c}{v}$$

$$1+\frac{v}{c}-1-\frac{c}{v}=-\frac{c}{vk^2}$$
$$k^2\left(\frac{v}{c}-\frac{c}{v}\right)=-\frac{c}{v}$$
$$k^2=\frac{-c/v}{v/c-c/v}=\frac{c/v}{c/v-v/c}=\frac{1}{1-\left[(v/c)/(c/v)\right]}=\frac{1}{1-v^2/c^2}$$

Thus, the function k becomes

$$k=\frac{1}{\sqrt{1-v^2/c^2}} \tag{1.48}$$

Substituting this value of k into equation 1.41 gives the first of the new transformation equations, namely

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}} \tag{1.49}$$

Equation 1.49 gives the position x' of the body in the moving coordinate system in terms of its position x and velocity v at the time t in the stationary coordinate system. Before discussing its physical significance let us also substitute k into the time equation 1.44, that is,

$$t'=\frac{t}{\sqrt{1-v^2/c^2}}+\left(\frac{1-(1)^2/\left(\sqrt{1-v^2/c^2}\right)^2}{v/\sqrt{1-v^2/c^2}}\right)x$$

Simplifying,

$$t'=\frac{t}{\sqrt{1-v^2/c^2}}+\left(1-\frac{1}{1-v^2/c^2}\right)\left(\frac{x}{v}\right)\sqrt{1-v^2/c^2}$$
$$=\frac{t}{\sqrt{1-v^2/c^2}}+\frac{\left(1-v^2/c^2-1\right)x}{\left(1-v^2/c^2\right)v}\sqrt{1-v^2/c^2}$$
$$=\frac{t}{\sqrt{1-v^2/c^2}}-\frac{v^2/c^2}{\sqrt{1-v^2/c^2}}\frac{x}{v}$$

and the second transformation equation becomes

$$t'=\frac{t-xv/c^2}{\sqrt{1-v^2/c^2}} \tag{1.50}$$

These new transformation equations are called the **Lorentz transformations.**[2] The Lorentz transformation equations are summarized as

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{1.49}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.50}$$

Now that we have obtained the new transformation equations, we must ask what they mean. First of all, note that the coordinate equation for the position does look like the Galilean transformation for position except for the term $\sqrt{1 - v^2/c^2}$ in the denominator of the x'-term. If the velocity v of the reference frame is small compared to c, then $v^2/c^2 \approx 0$, and hence,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{x - vt}{\sqrt{1 - 0}} = x - vt$$

Similarly, for the time equation, if v is much less than c then $v^2/c^2 \approx 0$ and also $xv/c^2 \approx 0$. Therefore, the time equation becomes

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} = \frac{t - 0}{\sqrt{1 - 0}} = t$$

Thus, the Lorentz transformation equations reduce to the classical Galilean transformation equations when the relative speed between the observers is small as compared to the speed of light. This reduction of a new theory to an old theory is called the correspondence principle and was first enunciated as a principle by Niels Bohr in 1923. It states that any new theory in physics must reduce to the well-established corresponding classical theory when the new theory is applied to the special situation in which the less general theory is known to be valid.

Because of this reduction to the old theory, the consequences of special relativity are not apparent unless dealing with enormous speeds such as those comparable to the speed of light.

2 [2]These equations were named for H. A. Lorentz because he derived them before Einstein's theory of special relativity. However, Lorentz derived these equations to explain the negative result of the Michelson-Morley experiment. They were essentially empirical equations because they could not be justified on general grounds as they were by Einstein.

Example 1.4

The value of $1/\sqrt{1-v^2/c^2}$ *for various values of v.* What is the value of $1/\sqrt{1-v^2/c^2}$ for (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 mi/s, and (c) $v = 0.8c$. Take $c = 3.00 \times 10^8$ m/s in SI units. It will be assumed in all the examples of relativity that the initial data are known to whatever number of significant figures necessary to demonstrate the principles of relativity in the calculations.

Solution

a. The speed v = (1610 km/hr)(1 hr/3600 s) = 0.447 km/s = 447 m/s. Hence,

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\left(447\text{ m/s}\right)^2/\left(3.00\times10^8\text{ m/s}\right)^2}}$$
$$= \frac{1}{\sqrt{1-2.22\times10^{-12}}}$$

This can be further simplified by the binomial expansion as

$$(1 - x)^n = 1 - nx$$

and hence,

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1-\left(\frac{1}{2}\right)2.22\times10^{-12} = 1.00000000000111 = 1$$

That is, the value is so close to 1 that we cannot determine the difference.

b. The velocity v = 1610 km/s, gives a value of

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(1.61\times10^6\text{ m/s})^2/(3.00\times10^8\text{ m/s})^2}}$$
$$= \frac{1}{\sqrt{1-2.88\times10^{-5}}}$$
$$= \frac{1}{\sqrt{0.99997}} = \frac{1}{0.99999}$$
$$= 1.00001$$

Now 1610 km/s is equal to 3,600,000 mph. Even though this is considered to be an enormous speed, far greater than anything people are now capable of moving at (for example, a satellite in a low earth orbit moves at about 18,000 mph, and the velocity of the earth around the sun is about 68,000 mph), the effect is still so small that it can still be considered to be negligible.

c. For a velocity of 0.8c the value becomes

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(0.8c)^2/c^2}} = \frac{1}{\sqrt{1-0.64}}$$
$$= \frac{1}{\sqrt{0.36}} = \frac{1}{0.600}$$
$$= 1.67$$

Thus, at the speed of eight-tenths of the speed of light the factor becomes quite significant.

Go to Interactive Example

The Lorentz transformation equations point out that space and time are intimately connected. Notice that the position x' not only depends on the position x but also depends on the time t, whereas the time t' not only depends on the time t but also depends on the position x. *We can no longer consider such a thing as absolute time, because time now depends on the position of the observer. That is, all time must be considered relative. Thus, we can no longer consider space and time as separate entities. Instead there is a union or fusion of space and time into the single reality called spacetime.* That is, space by itself has no meaning; time by itself has no meaning; only spacetime exists. The coordinates of an event in four-dimensional spacetime are (x, y, z, t). We will say more about spacetime in chapter 2.

An interesting consequence of this result of special relativity is its effects on the fundamental quantities of physics. In general physics we saw that the world could be described in terms of three fundamental quantities--space, time, and matter. It is now obvious that there are even fewer fundamental quantities. *Because space and time are fused into spacetime, the fundamental quantities are now only two, spacetime and matter.*

It is important to notice that the Lorentz transformation equations for special relativity put a limit on the maximum value of v that is attainable by a body, because if $v = c$,

$$x' = \frac{x-vt}{\sqrt{1-v^2/c^2}} = \frac{x-vt}{\sqrt{1-c^2/c^2}}$$
$$= \frac{x-vt}{0}$$

Since division by zero is undefined, we must take the limit as v approaches c. That is,

$$x' = \lim_{v \to c} \frac{x-vt}{\sqrt{1-v^2/c^2}} = \infty$$

and similarly

$$t' = \lim_{v \to c} \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} = \infty$$

That is, for $v = c$, the coordinates x' and t' are infinite, or at least undefinable. If $v > c$ then $v^2/c^2 > 1$ and $1 - v^2/c^2 < 1$. This means that the number under the square root sign is negative and the square root of a negative quantity is imaginary. Thus x' and t' become imaginary quantities. Hence, according to the theory of special relativity, no object can move at a speed equal to or greater than the speed of light.

Example 1.5

Lorentz transformation of coordinates. A man on the earth measures an event at a point 5.00 m from him at a time of 3.00 s. If a rocket ship flies over the man at a speed of 0.800*c*, what coordinates does the astronaut in the rocket ship attribute to this event?

Solution

The location of the event, as observed in the moving rocket ship, found from equation 1.49, is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{5.00 \text{ m} - (0.800)(3.00 \times 10^8 \text{ m/s})(3.00 \text{ s})}{\sqrt{1 - (0.800c)^2/c^2}}$$

$$= -1.20 \times 10^9 \text{ m}$$

This distance is quite large because the astronaut is moving at such high speed. The event occurs on the astronaut's clock at a time

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{3.00 \text{ s} - (0.800)(3.00 \times 10^8 \text{ m/s})(5.00 \text{ m})/(3.00 \times 10^8 \text{ m/s})^2}{\sqrt{1 - (0.800c)^2/c^2}}$$

$$= 5.00 \text{ s}$$

Go to Interactive Example

The inverse Lorentz transformation equations from the moving system to the stationary system can be written down immediately by the use of the first postulate. That is, their form must be the same, but $-v$ is replaced by $+v$ and primes and unprimes are interchanged. Therefore, the inverse Lorentz transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.51}$$

$$y = y' \tag{1.52}$$

$$z = z' \tag{1.53}$$

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.54}$$

1.8 The Lorentz-Fitzgerald Contraction

Consider a rod at rest in a stationary coordinate system S on the earth, as in figure 1.13(a). What is the length of this rod when it is observed by an astronaut in the S' frame of reference, a rocket ship traveling at a speed v? One end of the rod is located at the point x_1, while the other end is located at the point x_2. *The length of this stationary rod, measured in the frame where it is at rest, is called its **proper length** and is denoted by L_0, where*

$$L_0 = x_2 - x_1 \tag{1.55}$$

What is the length of this rod as observed in the rocket ship? The astronaut must measure the coordinates x_1' and x_2' for the ends of the rod at the same time t' in his frame S'.

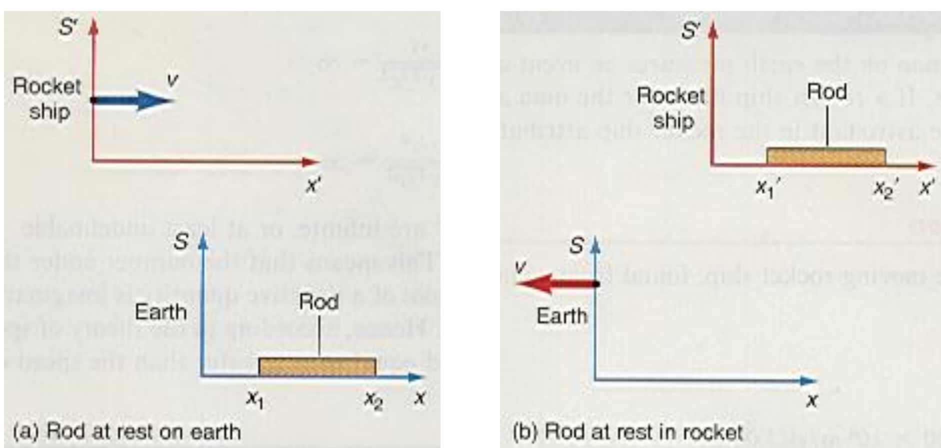


Figure 1.13 The Lorentz-Fitzgerald contraction.

The measurement of the length of any rod in a moving coordinate system must always be measured simultaneously in that coordinate system or else the ends of the rod will have moved during the measurement process and we will not be measuring the true length of the object. An often quoted example for the need of simultaneous measurements of length is the measurement of a fish in a tank. If the tail of the fish is measured first, and the head some time later, the fish has moved to the left and we have measured a much longer fish than the one in the tank, figure 1.14(a). If the head of the fish is measured first, and then the tail, the fish appears smaller than it is, figure 1.14(b). If, on the other hand, the head and tail are measured simultaneously we get the actual length of the fish, figure 1.14(c).

In a coordinate system where the rod or body is at rest, simultaneous measurements are not necessary because we can measure the ends at any time, since the rod is always there in that place and its ends never move. When the values of the coordinates of the end of the bar, x_1' and x_2', are measured at the time t', the values of x_1 and x_2 in the earth frame S are computed by the Lorentz transformation. Thus,

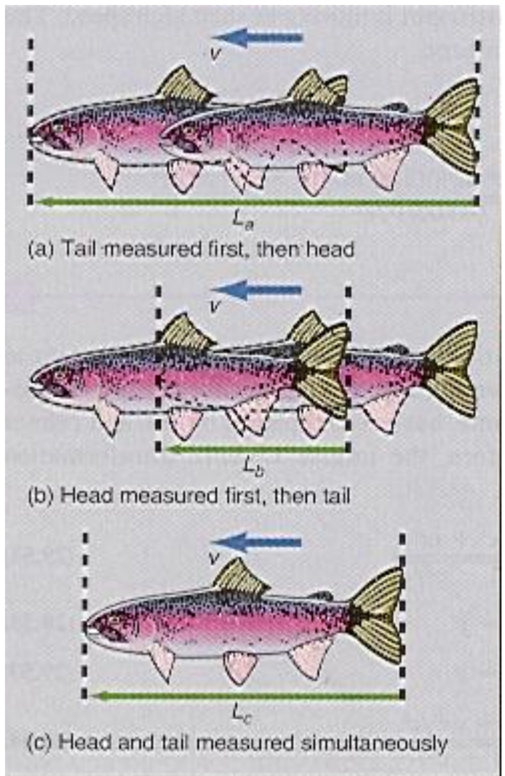


Figure 1.14 Measurement of the length of a moving fish.

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.56}$$

while

$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.57}$$

The length of the rod L_0, found from equations 1.55, 1.56, and 1.57, is

$$L_0 = x_2 - x_1 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} - \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{x_2' + vt' - x_1' - vt'}{\sqrt{1 - v^2/c^2}}$$

$$L_0 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} \tag{1.58}$$

Let us designate L as the length of the rod as measured in the moving rocket frame S', that is,

$$L = x_2' - x_1' \tag{1.59}$$

Then equation 1.58 becomes

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or

$$L = L_0\sqrt{1 - v^2/c^2} \tag{1.60}$$

Because v is less than c, the quantity $\sqrt{1 - v^2/c^2} < 1$, which means that $L < L_0$. *That is, the rod at rest in the earth frame would be measured in the rocket frame to be smaller than it is in the earth frame. From the point of view of the astronaut in the rocket, the rocket is at rest and the rod in the earth frame is moving toward him at a velocity v. Hence, the astronaut considers the rod to be at rest in a moving frame, and he then concludes that a moving rod contracts, as given by equation 1.60.* That is, if the rod is a meterstick, then its proper length in the frame where it is at rest is L_0 = 1.00 m = 100 cm. If the rocket is moving at a speed of 0.8c, then the length as observed from the rocket ship is

$$L = L_0\sqrt{1 - v^2/c^2} = 1.00\text{ m}\sqrt{1 - (0.8c)^2/c^2} = 0.600\text{ m}$$

Thus, the astronaut says that the meterstick is only 60.0 cm long. *This contraction of length is known as the **Lorentz-Fitzgerald contraction*** because it was derived earlier by Lorentz and Fitzgerald. However Lorentz and Fitzgerald attributed this effect to the ether. But since the ether does not exist, this effect cannot be attributed to it. It was Einstein's derivation of these same equations by the postulates of relativity that gave them real meaning.

This length contraction is a reciprocal effect. If there is a rod (a meterstick) at rest in the rocket S' frame, figure 1.13(b), then the astronaut measures the length of that rod by measuring the ends x_1' and x_2' at any time. The length of the rod as observed by the astronaut is

$$L_0 = x_2' - x_1' \tag{1.61}$$

This length is now the proper length L_0 because the rod is at rest in the astronaut's frame of reference. The observer on earth (S frame) measures the coordinates of the ends of the rod, x_1 and x_2, simultaneously at the time t. The ends of the rod x_2' and x_1' are computed by the earth observer by the Lorentz transformations:

$$x_2' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$$

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

Thus, the length of the rod becomes

$$\begin{aligned} L_0 = x_2' - x_1' &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - vt - x_1 + vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

But

$$x_2 - x_1 = L$$

the length of the rod as observed by the earth man. Therefore,

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

and

$$L = L_0 \sqrt{1 - v^2/c^2} \tag{1.62}$$

But this is the identical equation that was found before (equation 1.60). If L_0 is again the meterstick and the rocket ship is moving at the speed 0.800*c,* then the length L as observed on earth is 60.0 cm. *Thus the length contraction effect is reciprocal.* When the meterstick is at rest on the earth the astronaut thinks it is only 60.0 cm long. When the meterstick is at rest in the rocket ship the earthbound observer thinks it is only 60.0 cm long. Thus each observer sees the other's length as contracted. This reciprocity is to be expected. If the two observers do not agree that the other's stick is contracted, they could use this information to tell which stick is at rest and which is in motion--a violation of the principle of relativity. One thing that is important to notice is that in equation 1.60, L_0 is the length of the rod at rest in the earth or S frame of reference, whereas in equation 1.62, L_0 is the length of the rod at rest in the moving rocket ship (S' frame). L_0 *is always the length of the rod in the frame of reference where it is at rest.* It does not matter if the frame of reference is at rest or moving so long as the rod is at rest in that frame. This is why L_0 is always called its proper length.

The Lorentz-Fitzgerald contraction can be summarized by saying that the length of a rod in motion with respect to an observer is less than its length when measured by an observer who is at rest with respect to the rod. This contraction occurs only in the direction of the relative motion. Let us consider the size of this contraction.

Example 1.6

Length contraction. What is the length of a meterstick when it is measured by an observer moving at (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 miles/s, and (c) $v = 0.8c$. It is assumed in all these problems in relativity that the initial data are known to whatever number of significant figures necessary to demonstrate the principles of relativity in the calculations.

Solution

a. The speed v = (1610 km/hr)(1 hr/3600 s) = 0.447 km/s = 447 m/s. Take $c = 3.00 \times 10^8$ m/s in SI units. The length contraction, found from either equation 1.60 or equation 1.62, is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00\text{ m})\sqrt{1-\frac{(447\text{ m/s})^2}{(3.00\times 10^8\text{ m/s})^2}}$$
$$= (1.00\text{ m})\sqrt{1-2.22\times 10^{-12}}$$

This can be further simplified by the binomial expansion as

$$(1-x)^n = 1 - nx$$

and hence

$$\sqrt{1-\frac{v^2}{c^2}} = 1-\left(\frac{1}{2}\right)2.22\times 10^{-12} = 1-0.00000000000111$$
$$= 0.99999999999888$$

and

$$L = 1.00\text{ m}$$

Thus, at what might be considered the reasonably fast speed of 1000 mph, the contraction is so small that it is less than the width of one atom, and is negligible.

b. The contraction for a speed of 1610 km/s is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00\text{ m})\sqrt{1-\frac{(1.610\times 10^6\text{ m/s})^2}{(3.00\times 10^8\text{ m/s})^2}}$$
$$= (1.00\text{ m})\sqrt{0.99997}$$
$$= 0.99997\text{ m}$$

A speed of 1610 km/s is equivalent to a speed of 3,600,000 mph, which is an enormous speed, one man cannot even attain at this particular time. Yet the associated contraction is very small.

c. For a speed of $v = 0.8c$ the contraction is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$
$$= (1.00 \text{ m})\sqrt{1-\frac{(0.8c)^2}{c^2}} = (1.00 \text{ m})\sqrt{0.360}$$
$$= 0.600 \text{ m}$$

At speeds approaching the speed of light the contraction is quite significant. Table 1.1 gives the Lorentz contraction for a range of values of speed approaching the speed of light. Notice that as v increases, the contraction becomes greater and greater, until at a speed of $0.999999c$ the meterstick has contracted to a thousandth of a meter or 1 mm. Therefore, the effects of relativity do not manifest themselves unless very great speeds are involved. This is why these effects had never been seen or even anticipated when Newton was formulating his laws of physics. However, in the present day it is possible to accelerate charged particles, such as electrons and protons, to speeds very near the speed of light, and the relativistic effects are observed with such particles.

Table 1.1
The Lorentz Contraction and Time Dilation

Speed	$L = L_0\sqrt{1-\frac{v^2}{c^2}}$	$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$
$0.1c$	$0.995L_0$	$1.01\Delta t_0$
$0.2c$	$0.980L_0$	$1.02\Delta t_0$
$0.4c$	$0.917L_0$	$1.09\Delta t_0$
$0.6c$	$0.800L_0$	$1.25\Delta t_0$
$0.8c$	$0.602L_0$	$1.66\Delta t_0$
$0.9c$	$0.437L_0$	$2.29\Delta t_0$
$0.99c$	$0.141L_0$	$7.08\Delta t_0$
$0.999c$	$0.045L_0$	$22.4\Delta t_0$
$0.9999c$	$0.014L_0$	$70.7\Delta t_0$
$0.99999c$	$0.005L_0$	$224\Delta t_0$
$0.999999c$	$0.001L_0$	$707\Delta t_0$

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1.9 Time Dilation

Consider a clock at rest at the position x' in a moving coordinate system S' attached to a rocket ship. The astronaut sneezes and notes that he did so when his clock, located at x', reads a time t_1'. Shortly thereafter he sneezes again, and now notes that his clock indicates the time t_2'. The time interval between the two sneezes is

$$\Delta t' = t_2' - t_1' = \Delta t_0 \tag{1.63}$$

This interval $\Delta t'$ is set equal to Δt_0, and is called the ***proper time*** *because this is the time interval on a clock that is at rest relative to the observer.* The observer on earth in the S frame finds the time for the two sneezes to be

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$
$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

Thus, the time interval between the sneezes Δt, as observed by the earth man, becomes

$$\Delta t = t_2 - t_1 = \frac{t_2' + vx'/c^2 - t_1' - vx'/c^2}{\sqrt{1 - v^2/c^2}}$$
$$= \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

But $t_2' - t_1' = \Delta t_0$, by equation 1.63, therefore,

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}} \tag{1.64}$$

Notice that because $v < c$, $v^2/c^2 < 1$ and thus $\sqrt{1 - v^2/c^2} < 1$. Therefore,

$$\Delta t > \Delta t_0 \tag{1.65}$$

Equation 1.64 is the ***time dilation formula*** *and equation 1.65 says that the clock on earth reads a longer time interval Δt than the clock in the rocket ship Δt_0. Or as is sometimes said, moving clocks slow down.* Thus, if the moving clock slows down, a smaller time duration is indicated on the moving clock than on a stationary clock. *Hence, the astronaut ages at a slower rate than a person on earth.* The amount of this slowing down of time is relatively small as seen in example 1.7.

Example 1.7

Time dilation. A clock on a rocket ship ticks off a time interval of 1 hr. What time elapses on earth if the rocket ship is moving at a speed of (a) 1610 km/hr = 1000 mph, (b) 1610 km/s = 1000 mi/s, and (c) 0.8*c*?

Solution

a. The time elapsed on earth, found from equation 1.64, is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2 / c^2}}$$
$$= \frac{1 \text{ hr}}{\sqrt{1 - (447 \text{ m/s})^2 / (3.00 \times 10^8 \text{ m/s})^2}}$$
$$= 1 \text{ hr}$$

The difference between the time interval on the astronaut's clock and the time interval on the earthman's clock is actually about 4 ns. This is such a small quantity that it is effectively zero and the difference between the two clocks can be considered to be zero for a speed of 1600 km/s = 1000 mph.

b. The time elapsed for a speed of 1610 km/s is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2 / c^2}} = \frac{1 \text{ hr}}{\sqrt{1 - (1.61 \times 10^6 \text{ m/s})^2 / (3.00 \times 10^8 \text{ m/s})^2}}$$
$$= 1.0000144 \text{ hr}$$

Even at the relatively large speed of 1610 km/s = 3,600,000 mph, the difference in the clocks is practically negligible, that is a difference of 0.05 s in a time interval of 1 hr.

c. The time elapsed for a speed of 0.8*c* is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2 / c^2}} = \frac{1 \text{ hr}}{\sqrt{1 - (0.8c)^2 / (c)^2}}$$
$$= 1.66 \text{ hr}$$

Therefore, at very high speeds the time dilation effect is quite significant. Table 1.1 shows the time dilation for various values of v. As we can see, the time dilation effect becomes quite pronounced for very large values of v.

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It should be noted that the time dilation effect, like the Lorentz contraction, is also reciprocal. That is, a clock on the surface of the earth reads the proper time interval Δt_0 to an observer on the earth. An astronaut observing this earth clock assumes that he is at rest, but the earth is moving away from him at the velocity $-v$. Thus, he considers the earth clock to be the moving clock, and he finds that time on earth moves slower than the time on his rocket ship. This reciprocity of time dilation has led to the most famous paradox of relativity, called the twin paradox. (A paradox is an apparent contradiction.) The reciprocity of time dilation seems to be a contradiction when applied to the twins.

As an example, an astronaut leaves his twin sister on the earth as he travels, at a speed approaching the speed of light, to a distant star and then returns. According to the formula for time dilation, time has slowed down for the astronaut and when he returns to earth he should find his twin sister to be much older than he is. But by the first postulate of relativity, the laws of physics must be the same in all inertial coordinate systems. Therefore, the astronaut says that it is he who is the one at rest and the earth is moving away from him in the opposite direction. Thus, the astronaut says that it is the clock on earth that is moving and hence slowing down. He then concludes that his twin sister on earth will be younger than he is, when he returns. Both twins say that the other twin should be younger after the journey, and hence there seems to be a contradiction. How can we resolve this paradox?

With a little thought we can see that there is no contradiction here. The Lorentz transformations apply to inertial coordinate systems, that is, coordinates that are moving at a constant velocity with respect to each other. The twin on earth is in fact in an inertial coordinate system and can use the time dilation equation. The astronaut who returns home, however, is not in an inertial coordinate system. If the astronaut is originally moving at a velocity v, then in order for him to return home, he has to decelerate his spaceship to zero velocity and then accelerate to the velocity $-v$ to travel homeward. *During the deceleration and acceleration process the spaceship is not an inertial coordinate system, and we cannot justify using the time dilation formula that was derived on the basis of inertial coordinate systems.* Hence there is a very significant difference between the twin that stays home on the earth and the astronaut. Here again is that same conflict that occurs when we try to use an equation that was derived by using certain assumptions. When the assumptions hold, the equation is correct. When the assumptions do not hold, the equation no longer applies. In this example, the Lorentz transformation equations were derived on the assumption that two coordinate systems were moving with respect to each other at constant velocity. The astronaut is in an accelerated coordinate system when he turns around to come home. Hence, he is not in an inertial coordinate system and is not entitled to use the time dilation formula.[3] However, as correctly predicted by the earth twin, time has slowed down for the astronaut and when he returns to earth he should find his twin sister to be much older than he is.

We will consider a deeper insight into the slowing down of time in chapter 2 when we draw spacetime diagrams and discuss the general theory of relativity, and again in chapter 2 when we examine the gravitational red shift by the theory of the quanta.

1.10 Transformation of Velocities

We have seen that the Galilean transformation of velocities is incorrect when dealing with speeds at or near the speed of light. That is, velocities such as $V = c + v$ or $V = c - v$ are incorrect. Therefore, new transformation equations are needed for velocities. The necessary equations are found by the Lorentz equations. The components of the velocity of an object in a stationary coordinate system S are

$$V_x = \frac{dx}{dt} \tag{1.66}$$

$$V_y = \frac{dy}{dt} \tag{1.67}$$

$$V_z = \frac{dz}{dt} \tag{1.68}$$

whereas the components of the velocity of that same body, as observed in the moving coordinate system, S', are

$$V_x' = \frac{dx'}{dt'} \tag{1.69}$$

$$V_y' = \frac{dy'}{dt'} \tag{1.70}$$

$$V_z' = \frac{dz'}{dt'} \tag{1.71}$$

The transformation of the x-component of velocity is obtained as follows. The Lorentz transformation for the x' coordinate, equation 1.49, is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

The differential dx' becomes

3 [3]This also points out a flaw in the derivation of the Lorentz transformation equations. Starting with inertial coordinate systems, if there is any time dilation caused by the acceleration of the coordinate system to the velocity v, it cannot be determined in this way.

$$dx' = \frac{dx - v\,dt}{\sqrt{1 - v^2/c^2}} \tag{1.72}$$

The time interval dt' is found from the Lorentz transformation equation 1.50, as

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

Taking the time differential dt we get

$$dt' = \frac{dt - dx(v/c^2)}{\sqrt{1 - v^2/c^2}} \tag{1.73}$$

The transformation for V'_x, found from equations 1.69, 1.72, and 1.73, is

$$V'_x = \frac{dx'}{dt'} = \frac{(dx - v\,dt)/\sqrt{1 - v^2/c^2}}{[dt - dx(v/c^2)]/\sqrt{1 - v^2/c^2}} \tag{1.74}$$

Canceling out the square root term in both numerator and denominator, gives

$$V'_x = \frac{dx - v\,dt}{dt - dx(v/c^2)}$$

Dividing both numerator and denominator by dt, gives

$$V'_x = \frac{dx/dt - v\,dt/dt}{dt/dt - (dx/dt)(v/c^2)}$$

But $dx/dt = V_x$ from equation 1.66, and $dt/dt = 1$. Hence,

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x} \tag{1.75}$$

Equation 1.75 is the Lorentz transformation for the x-component of velocity. Notice that if v is very small, compared to c, then the term $(v/c^2)V_x$ approaches zero, and this equation reduces to the Galilean transformation equation 1.12 as would be expected for low velocities.

The y-component of the velocity transformation is obtained similarly. Thus,

$$V'_y = \frac{dy'}{dt'} = \frac{dy}{[dt - (v/c^2)dx]/\sqrt{1 - v^2/c^2}}$$

$$= \frac{dy\sqrt{1 - v^2/c^2}}{[dt - (v/c^2)dx]}$$

Dividing numerator and denominator by *dt* gives

$$V_y^{'} = \frac{(dy/dt)\sqrt{1 - v^2/c^2}}{dt/dt - (v/c^2)dx/dt}$$

and therefore

$$V_y^{'} = \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.76}$$

A similar analysis for the *z*-component of the velocity gives

$$V_z^{'} = \frac{V_z\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.77}$$

Note, that for v very much less than c, these equations reduce to the Galilean equations, $V_y' = V_y$ and $V_z' = V_z$, as expected.

The Lorentz velocity transformation equations are summarized as

$$V_x^{'} = \frac{V_x - v}{1 - (v/c^2)V_x} \tag{1.75}$$

$$V_y^{'} = \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.76}$$

$$V_z^{'} = \frac{V_z\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.77}$$

The inverse transformations from the S' frame to the S frame can be written down immediately by changing primes for nonprimes and replacing $-v$ by $+v$. Thus,

$$V_x = \frac{V_x' + v}{1 + (v/c^2)V_x'} \tag{1.78}$$

$$V_y = \frac{V_y'\sqrt{1 - v^2/c^2}}{1 + (v/c^2)V_x'} \tag{1.79}$$

$$V_z = \frac{V_z'\sqrt{1 - v^2/c^2}}{1 + (v/c^2)V_x'} \tag{1.80}$$

Example 1.8

Galilean transformation of velocities versus the Lorentz transformation of velocities. Two rocket ships are approaching a space station, each at a speed of $0.9c$, with respect to the station, as shown in figure 1.15. What is their relative speed according to (a) the Galilean transformation and (b) the Lorentz transformation?

Solution

a. According to the space station observer, the space station is at rest and the two spaceships are closing on him, as shown in figure 1.15. The spaceship to the right is approaching at a speed $V_x = -0.9c$, in the space station coordinate system. The spaceship to the left is considered to be a moving coordinate system approaching

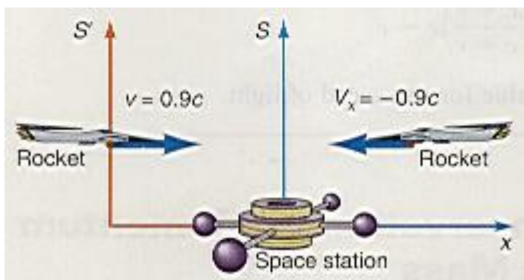

Figure 1.15 Galilean and Lorentz transformations of velocities.

with the speed $v = 0.9c$. The relative velocity according to the Galilean transformation, as observed in the moving spaceship to the left, is

$$\begin{aligned} V_x' &= V_x - v \\ &= -0.9c - 0.9c \\ &= -1.8c \end{aligned}$$

That is, the spaceship to the left sees the spaceship to the right approaching at a speed of $1.8c$. The minus sign means the velocity is toward the left in the S' frame of reference. Obviously this result is incorrect because the relative velocity is greater than c, which is impossible.

b. According to the Lorentz transformation the relative velocity of approach as observed by the S' spaceship, given by equation 1.75, is

$$V_x' = \frac{V_x - v}{1 - (v/c^2)V_x} = \frac{-0.9c - 0.9c}{1 - (0.9c/c^2)(-0.9c)} = \frac{-1.8c}{1 + 0.81c^2/c^2}$$

$$= \frac{-1.8c}{1.81} = -0.994c$$

Thus, the observer in the left-hand spaceship sees the right-hand spaceship approaching at the speed of 0.994c. The minus sign means that the speed is toward the left in the diagram. Notice that the relative speed is less than c as it must be.

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Example 1.9

Transformation of the speed of light. If a ray of light is emitted from a rocket ship moving at a speed v, what speed will be observed for that light on earth?

Solution

The speed of light from the rocket ship is $V'_x = c$. The speed observed on earth, found from equation 1.78, is

$$V_x = \frac{V_x' + v}{1 + (v/c^2)V'_x} = \frac{c + v}{1 + vc/c^2} = \frac{c + v}{1 + \left(\frac{v}{c}\right)}$$
$$= \frac{c+v}{(c+v)/c} = \left(\frac{c+v}{(c+v)}\right)c = c$$

Thus, all observers observe the same value for the speed of light.

1.11 The Law of Conservation of Momentum and Relativistic Mass

In section 1.3 we saw that momentum was conserved under a Galilean transformation. Does the law of conservation of momentum also hold in relativistic mechanics? Let us first consider the following perfectly elastic collision between two balls that are identical when observed in a stationary rest frame S in figure 1.16. The first ball m_A is thrown upward with the velocity V_y, whereas the second ball is thrown straight downward with the velocity $-V_y$. Thus the speed of each ball is the same. We assume that the original distance separating the two balls is small and the velocity V_y is relatively large, so that the effect of the acceleration due to gravity can be ignored. Applying the law of conservation of momentum to the collision, we obtain

momentum before collision = momentum after collision

$$m_A V_y - m_B V_y = m_B V_y - m_A V_y$$

Simplifying,

$$2m_A V_y = 2m_B V_y$$

or

$$m_A V_y = m_B V_y \tag{1.81}$$

Equation 1.81 also indicates that $m_A = m_B$, as originally stated.

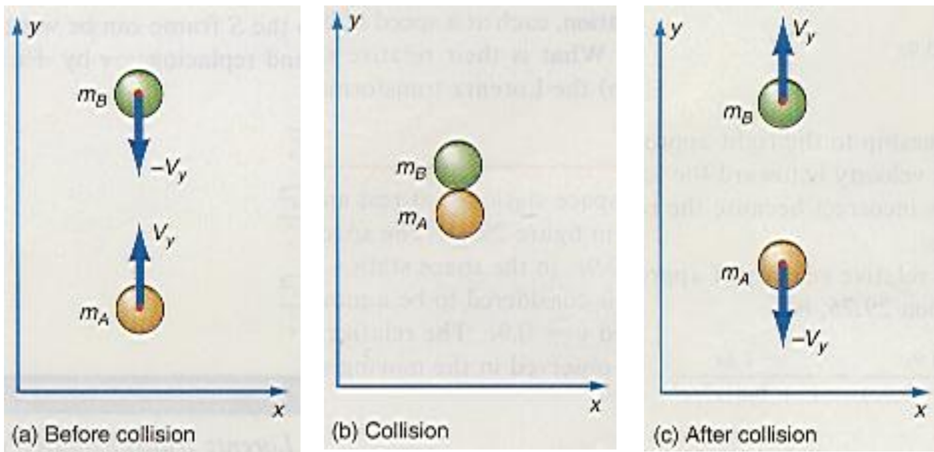

Figure 1.16 A perfectly elastic collision in a stationary frame of reference.

Let us now consider a similar perfectly elastic collision only now the ball B is thrown by a moving observer. The stationary observer is in the frame S and the moving observer is in the moving frame S', moving toward the left at the velocity $-v$, as shown in figure 1.17. In the stationary frame S, the observer throws a ball

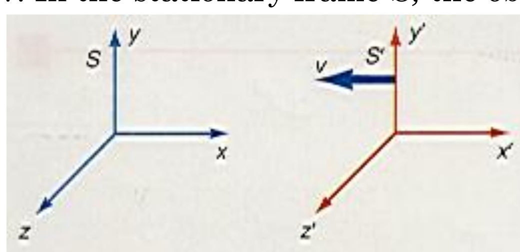

Figure 1.17 One observer is in rest frame and one in moving frame.

straight upward in the positive y-direction with the velocity V_y. The moving observer S' is on a truck moving to the left with the velocity $-v$. The moving observer throws an identical ball straight downward in the negative y-direction with the velocity $-u'$ in the moving frame of reference. In the stationary frame this velocity is observed as $-u$. We assume that both observers throw the ball with the same speed in their frames of reference. That is, the magnitude of the velocity V_y in the S frame is identical to the magnitude of the velocity u' in the S' frame. (As an example, let us assume that observer S throws the ball upward at a speed of 20 m/s and observer S' throws his ball downward at a speed of 20 m/s.) The two balls are exactly alike in that they have the same mass and size when they are at rest before the experiment starts.

After some practice, the experimenters are able to throw the balls such that a collision occurs. As observed from the S frame of reference on the ground, the collision appears as in figure 1.18(a). The mass m_A goes straight up, collides in a perfectly elastic collision with mass m_B, and is reflected with the velocity $-V_y$, since no energy, and hence speed, was lost in the collision. Ball B has a velocity component $-u$ straight downward (as seen by the S observer) but it is also moving in the negative x-direction with the velocity $-v$, the velocity of the truck and hence the velocity of the S' frame.

The y-component of the law of conservation of momentum, as observed in the rest frame S, can be written as

momentum before collision = momentum after collision

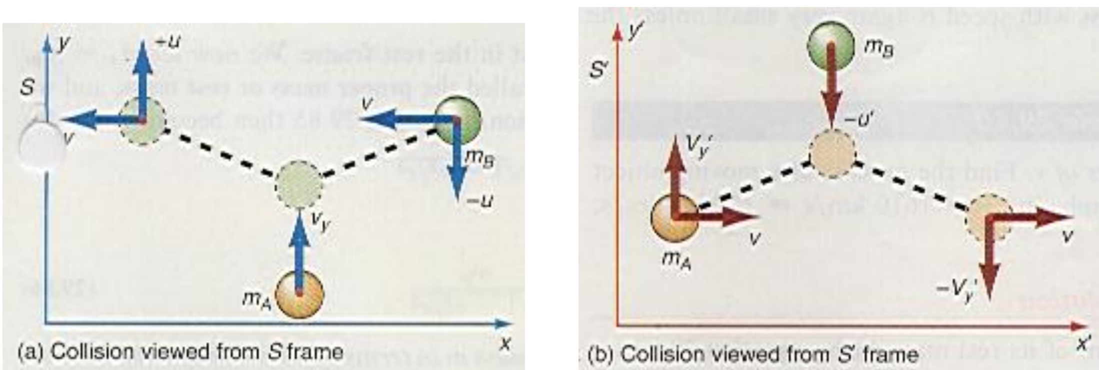

Figure 1.18 A perfectly elastic collision viewed from different frames of reference.

$$m_A V_y - m_B u = -m_A V_y + m_B u$$

Simplifying,

$$2m_A V_y = 2m_B u$$

or

$$m_A V_y = m_B u \tag{1.82}$$

But the velocity u in the stationary frame S is related to the velocity u' in the moving frame of reference S' through the velocity transformation, equation 1.79, as

$$u = \frac{u'\sqrt{1 - v^2/c^2}}{1 + (v/c^2)V_x'}$$

However, $V'x = 0$ in this experiment because m_B is thrown only in the y-direction. Hence, u becomes

$$u = u'\sqrt{1 - v^2/c^2} \tag{1.83}$$

Substituting u from equation 1.83 back into the law of conservation of momentum, equation 1.82, we obtain

$$m_A V_y = m_B u' \sqrt{1 - v^2 / c^2}$$

But recall that the initial speed of each ball was the same in each reference frame, that is, $V_y = u'$. Hence,

$$m_A V_y = m_B V_y \sqrt{1 - v^2 / c^2} \tag{1.84}$$

If we compare equation 1.84, for the conservation of momentum when one of the frames is in motion, with equation 1.81, for the conservation of momentum in a stationary frame, we see that the form of the equation is very different. Thus, in the form of equation 1.84, the law of conservation of momentum does not seem to hold. But the law of conservation of momentum is such a fundamental concept in physics that we certainly do not want to lose it in the description of relativistic mechanics. The law of conservation of momentum can be retained if we allow for the possibility that the moving mass changes its value because of that motion. That is, if both sides of equation 1.84 are divided by V_y we get

$$m_A = m_B \sqrt{1 - v^2 / c^2} \tag{1.85}$$

Now m_A is the mass of the ball in the stationary frame and m_B is the mass of the ball in the moving frame. If we consider the very special case where V_y is zero in the S frame, then the mass m_A is at rest in the rest frame. We now let $m_A = m_0$, the mass when it is at rest, henceforth called the **proper mass or rest mass,** and we let $m_B = m$, the mass when it is in motion. Equation 1.85 then becomes

$$m_0 = m \sqrt{1 - v^2 / c^2}$$

or, solving for m,

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}} \tag{1.86}$$

Equation 1.86 defines the ***relativistic mass*** *m in terms of its rest mass m_0.* Because the term $\sqrt{1 - v^2 / c^2}$ is always less than one, *the relativistic mass m, the mass of a body in motion at the speed v, is always greater than m_0, the mass of the body when it is at rest.* The variation of mass with speed is again very small unless the speed is very great.

Example 1.10

The relativistic mass for various values of v. Find the mass m of a moving object when (a) v = 1610 km/hr = 1000 mph, (b) v = 1610 km/s = 1000 miles/s, (c) $v = 0.8c$, and (d) $v = c$.

Solution

The relativistic mass m is found in terms of its rest mass m_0 by equation 1.86.

a. For v = 1610 km/hr = 447 m/s, we obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - (447 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}}$$

$$= m_0$$

Thus, at this reasonably high speed there is no measurable difference in the mass of the body.

b. For v = 1610 km/s,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - (1.610 \times 10^6 \text{ m/s})^2/(3.00 \times 10^8 \text{ m/s})^2}}$$

$$= \frac{m_0}{\sqrt{0.99997}}$$

$$= \frac{m_0}{0.99999}$$

$$= 1.00001 m_0$$

Thus, for a speed of 1610 km/s = 3,600,000 mph, a speed so great that macroscopic objects cannot yet attain it, the relativistic increase in mass is still practically negligible.

c. For $v = 0.8c$,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - (0.8c)^2/c^2}}$$

$$= 1.67 m_0$$

For the rather large velocity of 0.8c, the increase in mass is very significant. We should note that it is almost routine today to accelerate elementary particles to speeds approaching the speed of light and in all such cases this variation of mass with speed is observed.

d. For $v = c$,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{\sqrt{1 - c^2/c^2}} = \frac{m_0}{0}$$
$$= \infty$$

Thus, as a particle approaches the speed of light c, the mass of the particle approaches infinity. Since an infinite force and infinite energy would be required to move an infinite mass it is obvious that a particle of a finite rest mass m_0 can never be accelerated to the speed of light.

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The first of many experiments to verify the change in mass with speed was performed by A. H. Bucherer in 1909. Electrons were first accelerated by a large potential difference until they were moving at high speeds. They then entered a velocity selector. By varying the electric and magnetic field of the velocity selector, electrons with any desired velocity can be obtained by the equation $v = E/B$ These electrons were then sent through a uniform magnetic field B where they were deflected into a circular path. The centripetal force was set equal to the magnetic force, and we obtain

$$\frac{mv^2}{r} = qvB$$

Simplifying,

$$mv = qBr \tag{1.87}$$

But now we must treat the mass in equation 1.87 as the relativistic mass in equation 1.86. Thus, equation 1.87 becomes

$$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = qBr \tag{1.88}$$

Because B, r, and v could be measured in the experiment, the ratio of the charge of the electron q to its rest mass m_0, found from equation 1.88, is

$$\frac{q}{m_0} = \frac{v}{Br\sqrt{1-v^2/c^2}} \tag{1.89}$$

Bucherer's experiment confirmed equation 1.89 and hence the variation of mass with speed. Since 1909, thousands of experiments have been performed confirming the variation of mass with speed.

The variation of mass with speed truly points out the meaning of the concept of inertial mass as a measure of the resistance of matter to motion. As we can see with this relativistic mass, at higher and higher speeds there is a much greater resistance to motion and this is manifested as the increase in the mass of the body. The rest mass m_0 should probably be called the "quantity of matter" of a body since it is truly a measure of how much matter is present in the body, whereas the relativistic mass is the measure of the resistance of that quantity of matter to being put into motion.

With this new definition of relativistic mass the ***relativistic linear momentum*** *can now be defined as*

$$\mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1-v^2/c^2}} \tag{1.90}$$

The law of conservation of momentum now holds for relativistic mechanics just as it did for Newtonian mechanics. In fact, we can rewrite equation 1.84 as

$$m_0V_y = mV_y\sqrt{1-v^2/c^2}$$

Substituting for m from equation 1.86,

$$m_0V_y = \frac{m_0}{\sqrt{1-v^2/c^2}}V_y\sqrt{1-v^2/c^2}$$

Simplifying,

$$m_0V_y = m_0V_y \tag{1.91}$$

Hence, using the concept of relativistic mass, the same form of the equation for the law of conservation of momentum 1.91 is obtained as for the Newtonian case in equation 1.81. Thus, momentum is always conserved if the relativistic mass is used and the law of conservation of momentum is preserved for relativistic mechanics.

Newton's second law is still valid for relativistic mechanics, but only in the form

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{d}{dt}\left(\frac{m_0v}{\sqrt{1-v^2/c^2}}\right) \tag{1.92}$$

1.12 The Law of Conservation of Mass-Energy

As we have just seen, because the mass of an object varies with its speed, Newton's second law also changes for relativistic motion. We now ask what effect does this changing mass have on the kinetic energy of a moving body? How do we determine the kinetic energy of a body when its mass is changing with time? First let us recall how we determined the kinetic energy of a body of constant mass. The kinetic energy of a moving body was equal to the work done to put the body into motion, i.e.,

$$\text{KE} = W = \int dW = \int \mathbf{F} \bullet d\mathbf{s} = \int ma\, dx$$

However, since the acceleration $a = dv/dt$ this became

$$\text{KE} = \int madx = \int m\frac{dv}{dt}dx = \int mdv\frac{dx}{dt} = \int m\frac{dx}{dt}dv$$

Since $dx/dt = v$ the velocity of the moving body, the kinetic energy became

$$\text{KE} = \int_0^v mvdv = \left[\frac{mv^2}{2}\right]_0^v = \tfrac{1}{2}mv^2$$

Let us now see how this changes when we compute the kinetic energy relativisticly. The kinetic energy is again equal to the work done to put the body into motion. That is,

$$\text{KE} = W = \int dW = \int F\, dx$$

But Newton's second law is now written in the form $F = dp/dt$ and the kinetic energy becomes

$$\text{KE} = \int \frac{dp}{dt}dx = \int dp\frac{dx}{dt} = \int vdp \tag{1.93}$$

We cannot integrate equation 1.93 directly because $p = mv$ and hence dp is a function of v. We solve equation 1.93 by the standard technique of integrating it by parts, that is $\int vdp$ has the form $\int udV$ which has the standard solution

$$\int udV = Vu - \int Vdu \tag{1.94}$$

We let $u = v$ and hence $du = dv$, and $dV = dp$ hence $V = p$ and the integration becomes

$$\text{KE} = \int vdp = [pv]_0^v - \int_0^v pdv$$
$$\text{KE} = [(mv)v]_0^v - \int_0^v mvdv$$
$$\text{KE} = mv^2 - \int_0^v \frac{m_0 vdv}{\sqrt{1 - v^2/c^2}} \tag{1.95}$$

We integrate the second term in equation 1.95 by making the following substitutions. Let $x = v^2/c^2$, and hence $dx = (2vdv)/c^2$. Then $vdv = c^2dx/2$ and the integral in equation 1.95 becomes

$$I = m_0 \int_0^v \frac{vdv}{\sqrt{1-v^2/c^2}} = m_0 \int \frac{c^2dx}{2\sqrt{1-x}} \tag{1.96}$$

We now let

$$y = \sqrt{1-x}$$

Therefore

$$dy = 1/2(1-x)^{-1/2}(-dx)$$

and hence

$$dx = -2\sqrt{1-x}\,dy$$

and the second integral in equation 1.96 becomes

$$I = \frac{m_o c^2}{2}\int \frac{\left(-2\sqrt{1-x}\ \right)dy}{y} = \frac{m_o c^2}{2}\int \frac{(-2ydy)}{y}$$
$$= -m_0c^2 \int dy = -m_0c^2 y = -m_0c^2\sqrt{1-x} = \left[-m_0c^2\sqrt{1-v^2/c^2}\ \right]_0^v$$

and

$$I = \left[-m_0c^2\sqrt{1-v^2/c^2}\ \right]_0^v$$
$$I = m_0\left[-c^2\sqrt{1-v^2/c^2} + c^2\right]$$

Equation 1.95 therefore becomes

$$\text{KE} = mv^2 - m_0\left[-c^2\sqrt{1-v^2/c^2} + c^2\right]$$
$$\text{KE} = \frac{m_0}{\sqrt{1-v^2/c^2}}v^2 + m_0c^2\sqrt{1-v^2/c^2} - m_0c^2$$
$$\text{KE} = \frac{m_0v^2 + m_0c^2(1-v^2/c^2)}{\sqrt{1-v^2/c^2}} - m_0c^2$$
$$\text{KE} = \frac{m_0v^2 + m_0c^2 - m_0c^2v^2/c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$
$$\text{KE} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$$

and since $m_o/\sqrt{1-v^2/c^2} = m$, the relativistic kinetic energy becomes

$$\text{KE} = mc^2 - m_0c^2 \tag{1.97}$$

We can also write the relativistic mass m as

$$m = m_0 + \Delta m \tag{1.98}$$

That is, *the relativistic mass is equal to the rest mass plus the change in mass due to motion*. Substituting equation 1.98 back into equation 1.97, we have

$$\text{KE} = (m_0 + \Delta m)c^2 - m_0c^2$$

or

$$\text{KE} = (\Delta m)c^2 \tag{1.99}$$

Thus, relativistically, the kinetic energy of a body is equal to the change in mass of the body caused by the motion times the velocity of light squared.

Notice that the left-hand side of either equation 1.97 or 1.99 represents an energy. Since the left-hand side of the equation is equal to the right-hand side of the equation, the right-hand side must also represent an energy. That is, *the product of a mass times the square of the speed of light must equal an energy. The total* ***relativistic energy*** *of a body is, therefore, defined as*

$$E = mc^2 \tag{1.100}$$

We can rewrite equation 1.97 as

$$mc^2 = \text{KE} + m_0c^2$$

In view of the definition in equation 1.100, the total energy of a body is

$$E = \text{KE} + m_0c^2 \tag{1.101}$$

When a particle is at rest, its kinetic energy KE is equal to zero. Therefore, the total energy of the particle when it is at rest must be equal to m_0c^2. *The* ***rest mass energy*** *of a particle can then be defined as*

$$E_0 = m_0c^2 \tag{1.102}$$

Substituting equation 1.102 back into equation 1.101, we get

$$E = \text{KE} + E_0 \tag{1.103}$$

Equation 1.103 states that the total energy of a body is equal to its kinetic energy plus its rest mass energy. The result of these equations is that energy can manifest itself as mass, and mass can manifest itself as energy. In a sense, mass can be thought of as being frozen energy.

Example 1.11

Energy in a 1-kg mass. How much energy is stored in a 1.00-kg mass?

Solution

The rest mass energy of a 1.00-kg mass, given by equation 1.102, is

$$E_0 = m_0c^2$$
$$= (1.00 \text{ kg})(3.00 \times 10^8 \text{ m/s})^2$$
$$= 9.00 \times 10^{16} \text{ J}$$

This is an enormous amount of energy to be sure. It is about a thousand times greater than the energy released from the atomic bomb dropped on Hiroshima, and could supply 2.85 gigawatts of power for a period of one year.

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We have found the total relativistic kinetic energy to be given by equation 1.97 as

$$\text{KE} = mc^2 - m_0c^2$$

What does this relation for the kinetic energy reduce to at low speeds? The term for the mass m can be written as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0(1 - v^2/c^2)^{-1/2}$$

For relatively small velocities the term $(1 - v^2/c^2)^{-1/2}$ can be expanded by the binomial theorem, as

$$(1 - x)^n = 1 - nx$$
$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Substituting this back into the equation for the mass, we get

$$m = m_0\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$
$$= m_0 + \frac{1}{2}\frac{m_0v^2}{c^2}$$

Multiplying each term by c^2, gives

$$mc^2 = m_0c^2 + \frac{1}{2}m_0v^2$$

Replacing this into equation 1.97 for the kinetic energy gives

$$\text{KE} = mc^2 - m_0c^2 = m_0c^2 + \frac{1}{2}m_0v^2 - m_0c^2$$

and finally

$$KE = \frac{1}{2} m_0 v^2$$

Notice that the relativistic equation for the kinetic energy reduces to the classical form of the equation for the kinetic energy at low speeds as would be expected.

Example 1.12

Relativistic and classical kinetic energy. A 1.00-kg object is accelerated to a speed of 0.4*c*. Find its kinetic energy (a) relativistically and (b) classically.

Solution

a. The relativistic kinetic energy of the moving body, found from equation 1.97, is

$$\begin{aligned} KE &= mc^2 - m_0c^2 \\ &= \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 \\ &= \frac{1.00 \text{ kg}(3.0 \times 10^8 \text{ m/s})^2}{\sqrt{1 - (0.4c)^2/c^2}} - 1.00 \text{ kg}(3.00 \times 10^8 \text{ m/s})^2 \\ &= 8.20 \times 10^{15} \text{ J} \end{aligned}$$

b. The classical, and wrong, determination of the kinetic energy is

$$\begin{aligned} KE &= \frac{1}{2} mv^2 \\ &= \frac{1}{2}(1.00 \text{ kg})[(0.4)(3.00 \times 10^8 \text{ m/s})]^2 \\ &= 7.20 \times 10^{15} \text{ J} \end{aligned}$$

That is, if an experiment were performed to test these two results, the classical result would not agree with the experimental results, but the relativistic one would agree.

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When dealing with charged elementary particles, the kinetic energy can be found as the work that you must do in order to accelerate the particle up to the speed v, and is given as

$$KE = \text{work done} = qV$$

Example 1.13

Kinetic energy of an electron. An electron is accelerated through a uniform potential difference of 2.00×10^6 V. What is its kinetic energy as it leaves the electric field?

Solution

The kinetic energy, found from equation 23.74, is

$$\begin{aligned} \text{KE} &= qV \\ &= (1.60 \times 10^{-19}\ \text{C})(2.00 \times 10^6\ \text{V}) \\ &= 3.20 \times 10^{-13}\ \text{J} \end{aligned}$$

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It is customary in relativity and modern physics to express energies in terms of electron volts, abbreviated eV. The unit of energy called an electron volt is equal to the energy that an electron would acquire as it falls through a potential difference of 1 V. Hence,

$$\text{KE} = qV$$
$$1\ \text{eV} = (1.60 \times 10^{-19}\ \text{C})(1.00\ \text{V})$$
$$1\ \text{eV} = 1.60 \times 10^{-19}\ \text{J} \tag{1.104}$$

Thus, the electron volt is also a unit of energy.

Now we can express the KE in example 1.13 in electron volts as

$$\begin{aligned} \text{KE} &= \left(3.20 \times 10^{-13}\ \text{J}\right)\left(\frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{J}}\right) \\ &= 2.00 \times 10^6\ \text{eV} \end{aligned}$$

For larger quantities of energy the following units of energy are used:

$$1\ \text{kilo electron volt} = 1\ \text{keV} = 10^3\ \text{eV}$$
$$1\ \text{mega electron volt} = 1\ \text{MeV} = 10^6\ \text{eV}$$
$$1\ \text{giga electron volt} = 1\ \text{GeV} = 10^9\ \text{eV}$$
$$1\ \text{tera electron volt} = 1\ \text{TeV} = 10^{12}\ \text{eV}$$

Hence, the energy in example 1.13 can be expressed as

$$\text{KE} = 2.00\ \text{MeV}$$

By far, the greatest implication of equations 1.100 and 1.102 is that mass and energy must be considered as a manifestation of the same thing. *Thus, mass and* *energy are not independent quantities, just as we found that space and time are no longer independent quantities. Just as space and time are fused into spacetime, we must now fuse the separate concepts of mass and energy into one concept called mass-energy. What was classically considered as two separate laws, namely the law of conservation of mass and the law of conservation of energy must now be considered as one single law --* ***the law of conservation of mass-energy.*** *That is, mass can be created or destroyed as long as an equal amount of energy vanishes or appears, respectively.*

Because mass and energy can be equated it is sometimes desirable to express the mass of a particle in terms of energy units. Let us start by *defining an atomic unit of mass, called the unified mass unit, and defined as one-twelfth of the mass of the carbon 12 atom.* Recall that the mass of a molecule is given by

$$m = \frac{M}{N_{\mathbf{A}}}$$

where M is the molecular mass of the molecule and $N_{\mathbf{A}}$ is Avogadro's number. For a single atom the molecular mass is replaced by its atomic mass and the mass of a single atom is given by

$$m = \frac{\text{atomic mass}}{N_{\mathbf{A}}}$$

Thus, we define the *unified mass unit,* u, as

$$1\text{ u} = \frac{1}{12}m_{\mathbf{C}} = \frac{1}{12}\,\frac{12\text{ kg/kilomole}}{(6.0221367\times10^{26}\text{ molecules/kilomole})}$$

$$1\text{ u} = 1.660540\times10^{-27}\text{ kg} \tag{1.105}$$

To express this mass unit in terms of energy, we use equation 1.102 as

$$\begin{aligned} E_0 &= m_0c^2 \\ &= (1\text{ u})(c^2) \\ &= \frac{(1.660540\times10^{-27}\text{ kg})(2.997925\times10^{8}\text{ m/s})^2}{\left(1.60219\times10^{-19}\ \dfrac{\text{J}}{\text{eV}}\right)\left(\dfrac{10^6\text{ eV}}{\text{MeV}}\right)} \\ &= 931.493\text{ MeV} \end{aligned}$$

More significant figures have been used in this calculation than has been customary in this book. The additional accuracy is necessary because of the small quantities that are dealt with. Hence, a unified mass unit u has an energy equivalent of 931.493 MeV, that is,

$$1\text{ u} = 931.493\text{ MeV} \tag{1.106}$$

The masses of some of the elementary particles in terms of unified mass units and MeVs are given as

$$\text{rest mass of proton} = m_{\mathbf{p}} = 1.00726 \text{ u} = 938.256 \text{ MeV}$$
$$\text{rest mass of neutron} = m_{\mathbf{n}} = 1.00865 \text{ u} = 939.550 \text{ MeV}$$
$$\text{rest mass of electron} = m_{\mathbf{e}} = 0.00055 \text{ u} = 0.511006 \text{ MeV}$$
$$\text{rest mass of deuteron} = m_{\mathbf{d}} = 2.01410 \text{ u} = 1875.580 \text{ MeV}$$

Example 1.14

The energy of the deuteron. Deuterium is an isotope of hydrogen whose nucleus, called a *deuteron,* consists of a proton and a neutron. Find the sum of the rest mass energies of the proton and the neutron, and compare it with the rest mass energy of the deuteron.

Solution

The sum of the rest mass energy of the proton and neutron is

$$m_{\mathbf{p}} + m_{\mathbf{n}} = 938.26 \text{ MeV} + 939.55 \text{ MeV} = 1877.81 \text{ MeV}$$

The actual rest mass of the deuteron is $m_{\mathbf{d}}$ = 1875.58 MeV. Thus, the sum of the masses of the individual proton and neutron is greater than the mass of the deuteron itself. The difference in mass is

$$\begin{aligned}\Delta m &= (m_{\mathbf{p}} + m_{\mathbf{n}}) - m_{\mathbf{d}} \\ &= 1877.81 \text{ MeV} - 1875.58 \text{ MeV} \\ &= 2.23 \text{ MeV}\end{aligned}$$

That is, some mass Δm and hence energy is lost in combining the proton and the neutron. The lost energy that binds the proton and neutron together is called the binding energy of the system. This is the amount of energy that must be supplied to break up the deuteron.

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A further and extremely important application of mass-energy conversions occurs in the fusion of light atoms into heavier atoms. The most famous of such fusion processes is the conversion of hydrogen to helium in the sun and in the hydrogen bomb. An extremely simplified version of the process can be obtained by considering the mass of helium as consisting of two protons, two neutrons, and two electrons. The atomic mass of helium, as determined by the rest masses of its constituents, is

$$
\begin{aligned}
m_{He} &= 2m_p + 2m_n + 2m_e \\
&= 2(938.256 \text{ MeV}) + 2(939.550 \text{ MeV}) + 2(0.511006 \text{ MeV}) \\
&= 1876.512 \text{ MeV} + 1879.100 \text{ MeV} + 1.0220 \text{ MeV} \\
&= 3756.634 \text{ MeV}
\end{aligned}
$$

If this value is compared to the atomic mass of helium from the table of elements we find

$$
\begin{aligned}
\text{Atomic mass of He} &= (4.002603\text{u})\left(\frac{931.493\,\text{MeV}}{\text{u}}\right) \\
&= 3728.397 \text{ MeV}
\end{aligned}
$$

Hence, helium is lighter than the sum of its constituent parts. The difference in mass between helium and its constituent parts is

$$
\begin{aligned}
\Delta m &= 3756.634 \text{ MeV} - 3728.397 \text{ MeV} \\
&= 28.237 \text{ MeV}
\end{aligned}
$$

Thus, 28.237 MeV of energy is given off for each atom of helium formed. For the formation of 1 mole of helium, there are 6.02×10^{23} atoms. Hence, the total energy released per mole of helium formed is

$$
\begin{aligned}
\frac{\text{Energy released}}{\text{mole}} &= \left(28.237\frac{\text{MeV}}{\text{atom}}\right)\left(6.02\times 10^{23}\frac{\text{atoms}}{\text{mole}}\right) \\
&= \left(1.70\times 10^{25}\text{ MeV}\right)\left(\frac{10^{6}\text{ eV}}{\text{MeV}}\right)\left(\frac{1.60\times 10^{-19}\text{ J}}{\text{eV}}\right) \\
&= 2.73 \times 10^{12} \text{ J}
\end{aligned}
$$

Hence, in the formation of 1 mole of helium, a mass of only 4 g, 2,730,000,000,000 J of energy are released. This monumental amount of energy, which comes from the conversion of mass into energy, is continually being released by the sun. This fusion process is also the source of energy in the hydrogen bomb.

Example 1.15

A high-speed electron. An electron is accelerated from rest through a potential difference of 3.00×10^5 V. Find (a) the kinetic energy of the electron, (b) the total energy of the electron, (c) the speed of the electron, (d) the relativistic mass of the electron, and (e) the momentum of the electron.

Solution

a. The kinetic energy of the electron, found from equation 23.74, is

$$\text{KE} = \text{work done} = qV$$
$$\text{KE} = (1.60 \times 10^{-19}\ \text{C})(3.00 \times 10^{5}\ \text{V})$$
$$\text{KE} = \left(4.80 \times 10^{-14}\ \text{J}\right)\left(\frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{J}}\right)$$
$$\text{KE} = \left(3.00 \times 10^{5}\ \text{eV}\right)\left(\frac{1\ \text{MeV}}{10^{6}\ \text{eV}}\right)$$
$$= 0.300\ \text{MeV}$$

b. The rest mass energy of the electron is

$$E_0 = (m_0c^2)_{\text{electron}} = 0.511\ \text{MeV}$$

Thus, the total relativistic energy E, found from equation 1.101, is

$$E = \text{KE} + m_0c^2$$
$$= 0.300\ \text{MeV} + 0.511\ \text{MeV}$$
$$= 0.811\ \text{MeV}$$

c. To determine the speed of the electron, equation 1.97 is rearranged as

$$\text{KE} = mc^2 - m_0c^2$$
$$= \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)m_0c^2$$
$$\frac{1}{\sqrt{1 - v^2/c^2}} - 1 = \frac{\text{KE}}{m_0c^2}$$
$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\text{KE}}{m_0c^2} + 1 = \frac{0.300\ \text{MeV}}{0.511\ \text{MeV}} + 1 = 1.587$$
$$\sqrt{1 - v^2/c^2} = \frac{1}{1.587} = 0.630$$
$$1 - v^2/c^2 = (0.630)^2 = 0.397$$
$$v^2/c^2 = 1 - 0.397 = 0.603$$
$$v = \sqrt{0.603c^2}$$
$$v = 0.776c$$

Hence, the speed of the electron is approximately seven-tenths the speed of light.

d. To determine the relativistic mass of the electron, we use equation 1.86:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{9.11 \times 10^{-31} \text{ kg}}{\sqrt{1-(0.776c)^2/c^2}}$$
$$= 14.4 \times 10^{-31} \text{ kg}$$

The relativistic mass has increased by approximately 1.6 times the rest mass.

e. The momentum of the electron, found from equation 1.90, is

$$p = mv = \frac{m_0}{\sqrt{1-v^2/c^2}}v$$
$$= (14.4 \times 10^{-31} \text{ kg})(0.776)(3.00 \times 10^8 \text{ m/s})$$
$$= 3.35 \times 10^{-22} \text{ kg m/s}$$

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The Language of Physics

Relativity
The observation of the motion of a body by two different observers in relative motion to each other. At speeds approaching the speed of light, the length of a body contracts, its mass increases, and time slows down (p.).

Inertial coordinate system
A frame of reference that is either at rest or moving at a constant velocity (p.).

Galilean transformations
A set of classical equations that relate the motion of a body in one inertial coordinate system to that in a second inertial coordinate system. All the laws of classical mechanics are invariant under a Galilean transformation, but the laws of electromagnetism are not (p.).

Invariant quantity
A quantity that remains a constant whether it is observed from a system at rest or in motion (p.).

Ether
A medium that was assumed to pervade all space. This was the medium in which light was assumed to propagate (p.).

Michelson-Morley experiment
A crucial experiment that was performed to detect the presence of the ether. The

results of the experiment indicated that if the ether exists it cannot be detected. The assumption is then made that if it cannot be detected, it does not exist. Hence, light does not need a medium to propagate through. The experiment also implied that the speed of light in free space is the same everywhere regardless of the motion of the source or the observer (p.).

Special or Restricted Theory of Relativity
Einstein stated his special theory of relativity in terms of two postulates.

Postulate 1: The laws of physics have the same form in all inertial frames of reference.
Postulate 2: The speed of light in free space has the same value for all observers, regardless of their state of motion.

In order for the speed of light to be the same for all observers, space and time itself must change. The special theory is restricted to inertial systems and does not apply to accelerated systems (p.).

Lorentz transformations
A new set of transformation equations to replace the Galilean transformations. These new equations are derived by the two postulates of special relativity. These equations show that space and time are intimately connected. The effects of relativity only manifests itself when objects are moving at speeds approaching the speed of light (p.).

Proper length
The length of an object that is measured in a frame where the object is at rest (p.).

Lorentz-Fitzgerald contraction
The length of a rod in motion as measured by an observer at rest is less than its proper length (p.).

Proper time
The time interval measured on a clock that is at rest relative to the observer (p.).

Time dilation
The time interval measured on a moving clock is less than the proper time. Hence, moving clocks slow down (p.).

Proper mass or rest mass
The mass of a body that is at rest in a frame of reference (p.).

Relativistic mass
The mass of a body that is in motion. The relativistic mass is always greater than the rest mass of the object (p.).

Relativistic linear momentum
The product of the relativistic mass of a body and its velocity (p.).

Relativistic energy
The product of the relativistic mass of a body and the square of the speed of light. This total energy is equal to the sum of the kinetic energy of the body and its rest mass energy (p.).

Rest mass energy
The product of the rest mass and the square of the speed of light. Hence, mass can manifest itself as energy, and energy can manifest itself as mass (p.).

The law of conservation of mass-energy
Mass can be created or destroyed as long as an equal amount of energy vanishes or appears, respectively (p.).

Summary of Important Equations

Galilean transformation of coordinates

$$x = x' + vt \tag{1.1}$$
$$y = y' \tag{1.2}$$
$$z = z' \tag{1.3}$$
$$t = t' \tag{1.4}$$

Galilean transformation of velocities

$$v_x = v_x' + v \tag{1.11}$$
$$v_x' = v_x - v \tag{1.13}$$
$$v_y' = v_y \tag{1.14}$$
$$v_z' = v_z \tag{1.15}$$

Lorentz transformation equations of coordinates

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{1.49}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.50}$$

Inverse Lorentz transformation equations of coordinates

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.51}$$

$$y = y' \tag{1.52}$$
$$z = z' \tag{1.53}$$
$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.54}$$

Length contraction $$L = L_0\sqrt{1 - v^2/c^2} \tag{1.60}$$

Time dilation $$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \tag{1.64}$$

Lorentz transformation of velocities

$$V'_x = \frac{V_x - v}{1 - (v/c^2)V_x} \tag{1.75}$$
$$V'_y = \frac{V_y\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.76}$$
$$V'_z = \frac{V_z\sqrt{1 - v^2/c^2}}{1 - (v/c^2)V_x} \tag{1.77}$$

Relativistic mass $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{1.86}$$

Linear momentum $$\mathbf{p} = m\mathbf{v} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{1.90}$$

Newton's second law $$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \tag{1.92}$$

Relativistic kinetic energy $$\text{KE} = mc^2 - m_0c^2 \tag{1.97}$$
$$\text{KE} = (\Delta m)c^2 \tag{1.99}$$

Total relativistic energy $$E = mc^2 \tag{1.100}$$

Rest mass energy $$E_0 = m_0c^2 \tag{1.102}$$

Law of conservation of relativistic energy $$E = \text{KE} + E_0 \tag{1.103}$$

Electron volt $$1\ \text{eV} = 1.60 \times 10^{-19}\ \text{J} \tag{1.104}$$
$$\text{u} = 1.66 \times 10^{-27}\ \text{kg} \tag{1.105}$$

Unified mass unit $$\text{u} = 931.493\ \text{MeV} \tag{1.106}$$

Questions for Chapter 1

1. If you are in an enclosed truck and cannot see outside, how can you tell if you are at rest, in motion at a constant velocity, speeding up, slowing down, turning to the right, or turning to the left?

*2. Does a length contract perpendicular to its direction of motion?

*3. Lorentz explained the negative result of the Michelson-Morley experiment by saying that the ether caused the length of the telescope in the direction of motion to be contracted by an amount given by $L = L_0\sqrt{1 - v^2/c^2}$. Would this give a satisfactory explanation of the Michelson-Morley experiment?

4. If the speed of light in our world was only 100 km/hr, describe some of the characteristics of this world.

*5. Does time dilation affect the physiological aspects of the human body, such as aging? How does the body know what time is?

6. Are length contraction and time dilation real or apparent?

7. An elementary particle called a neutrino moves at the speed of light. Must it have an infinite mass? Explain.

*8. It has been suggested that particles might exist that are moving at speeds greater than *c*. These particles, which have never been found, are called tachyons. Describe how such particles might exist and what their characteristics would have to be.

9. In the equation for the total relativistic energy of a body, could there be another term for the potential energy of a body? Does a compressed spring, which has potential energy, have more mass than a spring that is not compressed?

*10. When helium is formed, the difference in the mass of helium and the mass of its constituents is given off as energy. When the deuteron is formed, the difference in mass is also given off as energy. Could the formation of deuterium be used as a source of commercial energy?

11. If the speed of light were infinite, what would the Lorentz transformation equations reduce to?

*12. Can you apply the Lorentz transformations to a reference frame that is moving in a circle?

Problems for Chapter 1

1.1 Introduction to Relative Motion

1. A projectile is thrown straight upward at an initial velocity of 25.0 m/s from an open truck at the same instant that the truck starts to accelerate forward at 5.00 m/s^2. If the truck is 4.00 m long, how far behind the truck will the projectile land?

2. A projectile is thrown straight up at an initial velocity of 25.0 m/s from an open truck that is moving at a constant speed of 10.0 m/s. Where does the projectile

land when (a) viewed from the ground (S frame) and (b) when viewed from the truck (S' frame)?

3. A truck moving east at a constant speed of 50.0 km/hr passes a traffic light where a car just starts to accelerate from rest at 2.00 m/s^2. At the end of 10.0 s, what is the velocity of the car with respect to (a) the traffic light and (b) with respect to the truck?

4. A woman is sitting on a bus 5.00 m from the end of the bus. If the bus is moving forward at a velocity of 7.00 m/s, how far away from the bus station is the woman after 10.0 s?

1.2 The Galilean Transformations of Classical Physics

5. The woman on the bus in problem 4 gets up and (a) walks toward the front of the bus at a velocity of 0.500 m/s. What is her velocity relative to the bus station? (b) The woman now walks toward the rear of the bus at a velocity of 0.500 m/s. What is her velocity relative to the bus station?

1.3 The Invariance of the Mechanical Laws of Physics under a Galilean Transformation

*6. Filling in the steps omitted in the derivation associated with figure 1.8, show that the law of conservation of momentum is invariant under a Galilean transformation.

*7. Show that the law of conservation of energy for a perfectly elastic collision is invariant under a Galilean transformation.

1.5 The Michelson-Morley Experiment

8. A boat travels at a speed V of 5.00 km/hr with respect to the water, as shown in figure 1.10. If it takes 90.0 s to cross the river and return and 95.0 s for the boat to go the same distance downstream and return, what is the speed of the river current?

1.7 The Lorentz Transformation

9. A woman on the earth observes a firecracker explode 10.0 m in front of her when her clock reads 5.00 s. An astronaut in a rocket ship who passes the woman on earth at $t = 0$, at a speed of $0.400c$ finds what coordinates for this event?

10. A clock in the moving coordinate system reads $t' = 0$ when the stationary clock reads $t = 0$. If the moving frame moves at a speed of $0.800c$, what time will the moving clock read when the stationary observer reads 15.0 hr on her clock?

*11. Use the Lorentz transformation to show that the equation for a light wave, equation 1.25, has the same form in a coordinate system moving at a constant velocity.

1.8 The Lorentz-Fitzgerald Contraction

12. The USS *Enterprise* approaches the planet Seti Alpha 5 at a speed of $0.800c$. Captain Kirk observes an airplane runway on the planet to be 2.00 km long. The air controller on the planet says that the runway on the planet is how long?

Diagram for problem 12.

13. The starship *Regulus* was measured to be 100 m long when in space dock. If it approaches a planet at a speed of $0.400c$, how long does it appear to an observer on the planet?

14. How fast must a 4.57 m car move in order to fit into a 30.5 cm garage? Could you park the car in this garage?

Diagram for problem 14.

15. A comet is observed to be 130 km long as it moves past an observer at a speed of $0.700c$. How long does the comet seem when it travels at a speed of $0.900c$ with respect to the observer?

16. A meterstick at rest makes an angle of 30.0^0 with the x-axis. Find the length of the meterstick and the angle it makes with the x'-axis for an observer moving parallel to the x-axis at a speed of $0.650c$.

1.9 Time Dilation

17. A particle is observed to have a lifetime of 1.50×10^{-6} s when it is at rest in the laboratory. (a) What is its lifetime when it is moving at $0.800c$? (b) How far will the particle move with respect to the moving frame of reference before it decays? (c) How far will the particle move with respect to the laboratory frame before it decays?

18. A stroboscope is flashing light signals at the rate of 2100 flashes/min. An observer in a rocket ship traveling toward the strobe light at $0.500c$ would see what flash rate?

19. A particle has a lifetime of 0.100 s when observed while it moves at a speed of $0.650c$ with respect to the laboratory. What is its lifetime in its rest frame?

1.10 Transformation of Velocities

20. A spaceship traveling at a speed of $0.600c$ relative to a planet launches a rocket backward at a speed of $0.500c$. What is the velocity of the rocket as observed from the planet?

21. The three electrons are moving at the velocities shown in the diagram. Find the relative velocities between (a) electrons 1 and 2, (b) electrons 2 and 3, and (c) electrons 1 and 3.

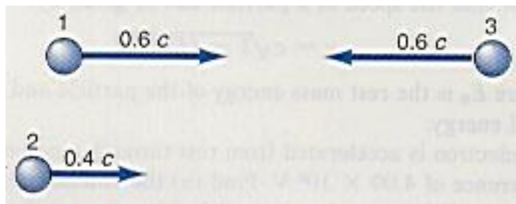

Diagram for problem 21.

1.11 The Law of Conservation of Momentum and Relativistic Mass

22. What is the mass of the following particles when traveling at a speed of $0.86c$: (a) electron, (b) proton, and (c) neutron?

23. Find the speed of a particle at which the mass m is equal to (a) 0.100 m_0, (b) 1.00 m_0, (c) 10.0 m_0, (d) 100 m_0, and (e) 1000 m_0.

24. Determine the linear momentum of an electron moving at a speed of $0.990c$.

25. How fast must a proton move so that its linear momentum is 8.08×10^{-19} kg m/s?

26. Compute the speed of a neutron whose total energy is 1.88×10^{-10} J.

1.12 The Law of Conservation of Mass-Energy

27. An isolated neutron is capable of decaying into a proton and an electron. How much energy is liberated in this process?

28. Since it takes 2.26×10^6 J to convert 1.00 kg of water to 1.00 kg of steam at 100 ^{0}C, what is the increase in mass of the steam?

29. What is the kinetic energy of a proton traveling at $0.800c$?

30. Through what potential difference must an electron be accelerated if it is to attain a speed of $0.800c$?

31. What is the total energy of a proton traveling at a speed of 2.50×10^8 m/s?

32. Calculate the speed of an electron whose kinetic energy is twice as large as its rest mass energy.

Additional Problems

33. If an ion-engine in a spacecraft can produce a continuous acceleration of 0.200 m/s^2, how long must the engine continue to accelerate if it is to reach the speed of 0.500c?

*34. The volume of a cube is V_0 in a frame of reference where it is at rest. Show that the volume observed in a moving frame of reference is given by

$$V = V_0\sqrt{1 - v^2 / c^2}$$

35. The distance to Alpha Centari, the closest star, is about 4.00 light years as measured from earth. What would this distance be as observed from a spaceship leaving earth at a speed of 0.500c? How long would it take to get there according to a clock on the spaceship and a clock on earth?

36. A muon is an elementary particle that is observed to have a lifetime of 2.00×10^{-6} s before decaying. It has a typical speed of 2.994×10^8 m/s. (a) How far can the muon travel before it decays? (b) These particles are observed high in our atmosphere, but with such a short lifetime how do they manage to get to the surface of the earth?

*37. Show that the formula for the density of a cube of material moving at a speed v is given by

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2 / c^2}}$$

*38. A proton is accelerated to a speed of 0.500c. Find its (a) kinetic energy, (b) total energy, (c) relativistic mass, and (d) momentum.

*39. Show that the speed of a particle can be given by

$$v = c\sqrt{1 - \left(E_0 / E\right)^2}$$

where E_0 is the rest mass energy of the particle and E is its total energy.

*40. An electron is accelerated from rest through a potential difference of 4.00×10^6 V. Find (a) the kinetic energy of the electron, (b) the total energy of the electron, (c) the velocity of the electron, (d) the relativistic mass, and (e) the momentum of the electron.

*41. From the solar constant, determine the total energy transmitted by the sun per second. How much mass is this equivalent to? If the mass of the sun is 1.99 $\times 10^{30}$ kg, approximately how long can the sun continue to radiate energy?

*42. A reference frame is accelerating away from a rest frame. Show that Newton's second law in the form $F = ma$ does not hold in the accelerated frame.

Interactive Tutorials

43. *Length contraction.* The length of a rod at rest is found to be $L_0 = 2.55$ m. Find the length L of the rod when observed by an observer in motion at a speed $v = 0.250c$.

44. *Time dilation.* A clock in a moving rocket ship reads a time duration $\Delta t_0 = 15.5$ hr. What time elapses, Δt, on earth if the rocket ship is moving at a speed $v = 0.355c$?

45. *Relative velocities.* Two spaceships are approaching a space station, as in figure 1.15. Spaceship 1 has a velocity of $0.55c$ to the left and spaceship 2 has a velocity of $0.75c$ to the right. Find the velocity of rocket ship 1 as observed by rocket ship 2.

46. *Relativistic mass.* A mass at rest has a value $m_0 = 2.55$ kg. Find the relativistic mass m when the object is moving at a speed $v = 0.355c$.

47. *Plot of length contraction and mass change versus speed.* The length of a rod at rest is $L_0 = 1.00$ m and its mass is $m_0 = 1.00$ kg. Find the length L and mass m of the rod as its speed v in the axial direction increases from $0.00c$ to $0.90c$, where c is the speed of light ($c = 3.00 \times 10^8$ m/s). Plot the results.

48. *An accelerated charged particle.* An electron is accelerated from rest through a potential difference $V = 4.55 \times 10^5$ V. Find (a) the kinetic energy of the electron, (b) the rest mass energy of the electron, (c) the total relativistic energy of the electron, (d) the speed of the electron, (e) the relativistic mass of the electron, and (f) the momentum of the electron.

Go to Interactive Tutorials

Chapter 2 Spacetime and General Relativity

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.
>
> H. Minkowski - "Space and Time"

2.1 Spacetime Diagrams

Shortly after Einstein published his special theory of relativity, Hermann Minkowski (1864-1909), a former instructor of Einstein, set about to geometrize relativity. He said that time and space are inseparable. In his words, "Nobody has ever noticed a place except at a time, or a time except at a place.... A point of space at a point of time, that is, a system of values of *x,y,z,t,* I will call a world-point. The multiplicity of all thinkable *x,y,z,t,* systems of values we will christen the world."[1]

To simplify the discussion, we will consider only one space dimension, namely the x-coordinate. *Any occurrence in spacetime will be called an event,* and is represented in the **spacetime diagram** of figure 2.1(a). This event might be the explosion of a firecracker, let us say. The location of this event is the *world point,* and it has the coordinates x and t. (Many authors of more advanced relativity books interchange the coordinates, showing the time axis in the vertical direction to emphasize that this is a different graph than a conventional plot of distance versus time. However, we will use the conventional graphical format in this book because it is already familiar to the student and will therefore make spacetime concepts easier to understand.)

Figure 2.1(b) is a picture of a **world line** of a particle at rest at the position x. *The graph shows that even though the particle is at rest in space, it is still moving through time.* Its x-coordinate is a constant because it is not moving through space, but its time coordinate is continually increasing showing its motion through time. Figure 2.1(c) represents a rod at rest in spacetime. The top line represents the world line of the end of the rod at x_2, whereas the bottom line represents the world line of the opposite end of the rod at x_1. Notice that the stationary rod sweeps out an area in spacetime. Figure 2.1(d) shows the world line of particle A moving at a constant velocity v_A and the world line of particle B moving at the constant velocity v_B. The slope of a straight line on an x versus t graph represents the velocity of the particle. The greater the slope, the greater the velocity. Since particle A has the greater slope it has the greater velocity, that is, $v_A > v_B$. If the velocity of a particle changes with time, its world line is no longer a straight line, but becomes curved, as shown in figure 2.1(e). *Thus, the world line of an accelerated particle is curved in spacetime.* Figure 2.1(f) is the world line of a mass attached to a spring that is executing simple harmonic motion. Note that the world line is curved everywhere

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1[1]"Space and Time," by H. Minkowski in *The Principle of Relativity,* Dover Publications.

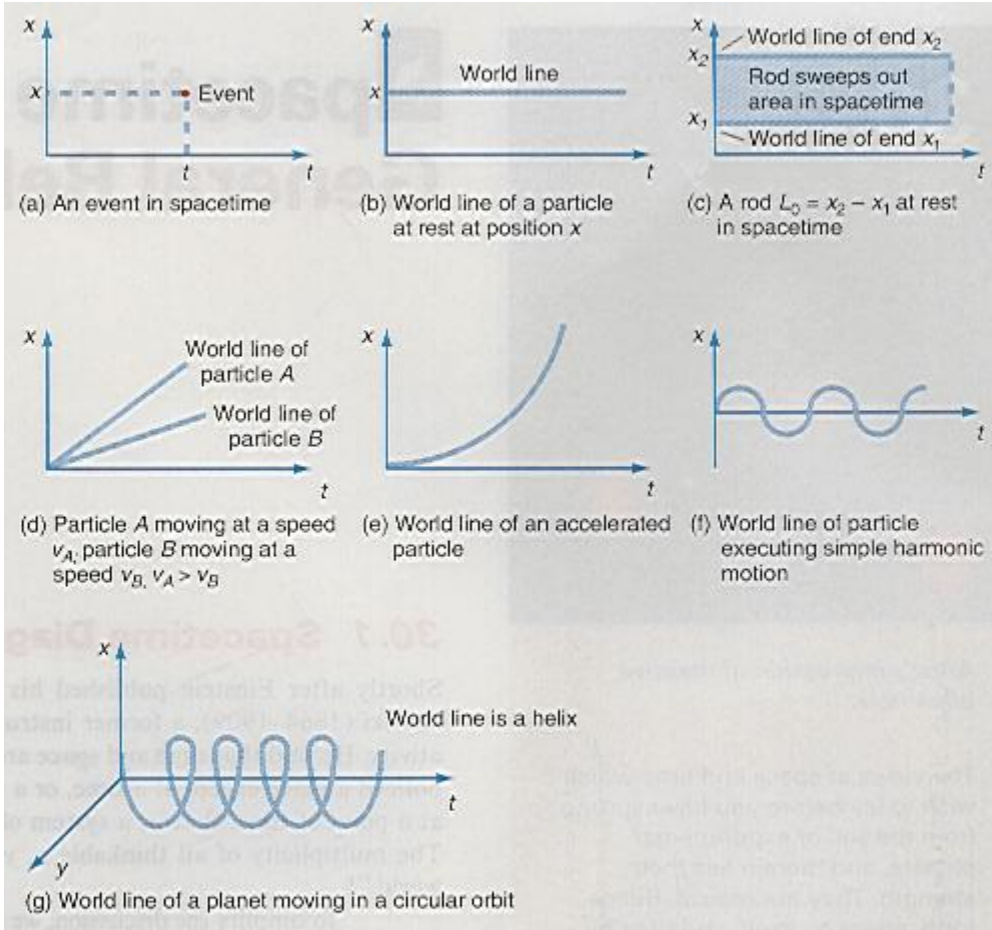

Figure 2.1 Spacetime diagrams.

indicating that this is accelerated motion. Figure 2.1(g) is a two-space dimensional picture of a planet in its orbit about the sun. The motion of the planet is in the x,y plane but since the planet is also moving in time, its world line comes out of the plane and becomes a helix. Thus, when the planet moves from position x, goes once around the orbit, and returns to the same space point x, it is not at the same position in spacetime. It has moved forward through time.

A further convenient representation in spacetime diagrams is attained by changing the time axis to τ, where

$$\tau = ct \tag{2.1}$$

In this representation, τ is actually a length. (The product of a velocity times the time is equal to a length.) The length τ is the distance that light travels in a particular time. If t is measured in seconds, then τ becomes a light second, which is the distance that light travels in 1 s, namely,

$$\tau = ct = \left(3.00 \times 10^8\ \frac{\text{m}}{\text{s}}\right)(1.00\ \text{s}) = 3.00 \times 10^8\ \text{m}$$

If t is measured in years, then τ becomes a light year, the distance that light travels in a period of time of 1 yr, namely,

$$\tau = ct = \left(3.00\times10^{8}\ \frac{\text{m}}{\text{s}}\right)(1\ \text{yr})\left(\frac{365\ \text{days}}{1\ \text{yr}}\right)\left(\frac{24\ \text{hr}}{1\ \text{day}}\right)\left(\frac{3600\ \text{s}}{1\ \text{hr}}\right)$$
$$= 9.47\times10^{15}\ \text{m} = 9.47\times10^{12}\ \text{km}$$

The light year is a unit of distance routinely used in astronomy.

With this new notation, we draw the spacetime diagram as shown in figure 2.2. A straight line on this diagram can still represent a velocity. However, since a

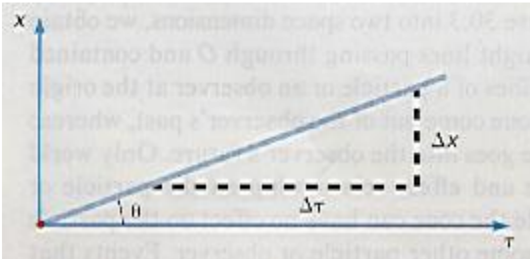

Figure 2.2 Changing the t-axis to a τ-axis.

velocity is given as

$$v = \frac{dx}{dt}$$

and since $\tau = ct$,

$$c\,dt = d\tau$$

or

$$dt = \frac{d\tau}{c}$$

Thus, the velocity becomes

$$v = \frac{dx}{dt} = \frac{dx}{d\tau/c} = \frac{c\,dx}{d\tau}$$

but $dx/d\tau$ is the slope of the line and is given by

$$\frac{dx}{d\tau} = \text{slope of line} = \tan\theta$$

Then the velocity on such a diagram is given by

$$v = c\tan\theta \tag{2.2}$$

As a special case in such a diagram, if $\theta = 45^0$, the $\tan 45^0 = 1$ and equation 2.2 becomes

$$v = c$$

Thus, on a spacetime diagram of x versus τ, a straight line at an angle of 45^0 represents the world line of a light signal.

If a source of light at the origin emits a ray of light simultaneously toward the right and toward the left, we represent it on a spacetime diagram as shown in figure 2.3. Line *OL* is the world line of the light ray emitted toward the right,

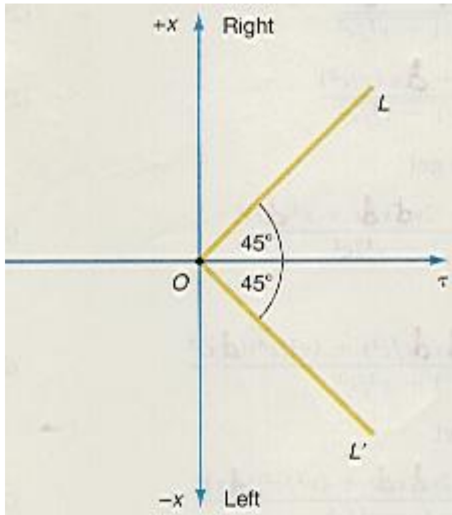

Figure 2.3 World lines of rays of light.

whereas *OL'* is the world line of the light ray emitted toward the left. Since the velocity of a particle must be less than c, the world line of any particle situated at *O* must have a slope less than 45^0 and is contained within the two light world lines *OL* and *OL'*. If the particle at *O* is at rest its world line is the τ-axis.

Example 2.1

The angle that a particle's world line makes as the particle moves through spacetime. If a particle moves to the right at a constant velocity of $c/2$, find the angle that its world line makes with the τ-axis.

Solution

Because the particle moves at a constant velocity through spacetime, its world line is a straight line. The angle that the world line makes with the τ-axis, found from equation 2.2, is

$$\theta = \tan^{-1}\frac{v}{c} \tag{2.3}$$

$$= \tan^{-1}\frac{c/2}{c} = \tan^{-1} 0.500$$

$$= 26.6^0$$

Notice that the world line for this particle is contained between the lines *OL* and *OL'*.

Go to Interactive Example

If we extend the diagram of figure 2.3 into two space dimensions, we obtain the **light cone** shown in figure 2.4. Straight lines passing through O and contained within the light cone are possible world lines of a particle or an observer

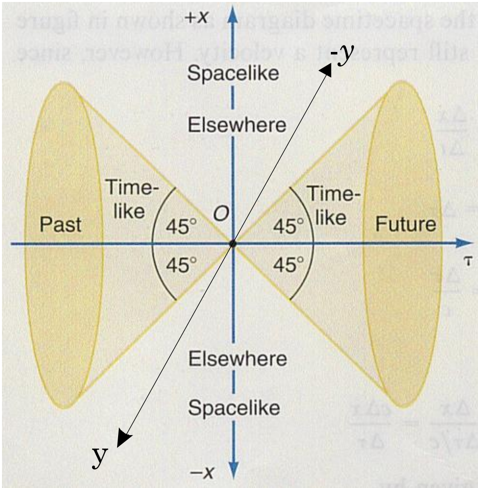

Figure 2.4 The light cone.

at the origin O. Any world lines inside the left-hand cone come out of the observer's past, whereas any world line inside the right-hand cone goes into the observer's future. Only world lines within the cone can have a cause and effect relationship on the particle or observer at O. World lines that lie outside the cone can have no effect on the particle or observer at O and are world lines of some other particle or observer. Events that we actually "see," lie on the light cone because we see these events by light rays. *World lines within the cone are sometimes called timelike because they are accessible to us in time. Events outside the cone are called spacelike because they occur in another part of space that is not accessible to us and hence is called elsewhere.*

2.2 The Invariant Interval

From what has been said so far, it seems as if everything is relative. *In the varying world of spacetime is there anything that remains a constant?* Is there some one single thing that all observers, regardless of their state of motion, can agree on? In the field of physics, we are always looking for some characteristic constants of motion. Recall from General Physics that when we studied the projectile motion of a particle in one dimension and saw that even though the projectile's position and velocity continually changed with time, there was one thing that always remained a constant, namely, the total energy of the projectile. In the same way we ask, isn't there a constant of the motion in spacetime? The answer is yes. *The constant value that all observers agree on, regardless of their state of motion, is called the* ***invariant interval.***

Let us now take the Lorentz transformation for the x-coordinate, equation 1.49

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

The differential dx' becomes

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} \tag{2.4}$$

Similarly, let us now take the Lorentz transformation for the t-coordinate, equation 1.50

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

Taking the time differential dt' we get

$$dt' = \frac{dt - dx(v/c^2)}{\sqrt{1 - v^2/c^2}} \tag{2.5}$$

Let us square each of these transformation equations to get

$$(dx')^2 = \frac{(dx)^2 - 2vdxdt + v^2(dt)^2}{1 - v^2/c^2} \tag{2.6}$$

and

$$(dt')^2 = \frac{(dt)^2 - (2vdxdt/c^2) + (v^2/c^4)(dx)^2}{1 - v^2/c^2} \tag{2.7}$$

Let us multiply equation 2.7 by c^2 to get

$$c^2(dt')^2 = \frac{c^2(dt)^2 - 2vdxdt + (v^2/c^2)(dx)^2}{1 - v^2/c^2} \tag{2.8}$$

Let us now subtract equation 2.6 from equation 2.8 to get

$$c^2(dt')^2 - (dx')^2 = \frac{c^2(dt)^2 - 2vdxdt + (v^2/c^2)(dx)^2}{1 - v^2/c^2} - \frac{(dx)^2 - 2vdxdt + v^2(dt)^2}{1 - v^2/c^2}$$

$$= \frac{c^2(dt)^2 - v^2(dt)^2 + (v^2/c^2)(dx)^2 - (dx)^2}{1 - v^2/c^2}$$

$$= \frac{(c^2 - v^2)(dt)^2 - (1 - v^2/c^2)(dx)^2}{1 - v^2/c^2}$$

$$c^2(dt')^2 - (dx')^2 = \frac{c^2(1 - v^2/c^2)(dt)^2 - (1 - v^2/c^2)(dx)^2}{1 - v^2/c^2}$$

Dividing each term on the right by $1 - v^2/c^2$ gives

$$c^2(dt')^2 - (dx')^2 = c^2(dt)^2 - (dx)^2 \tag{2.9}$$

Equation 2.9 shows that the quantity $c^2(dt)^2 - (dx)^2$ as measured by the S observer is equal to the same quantity $c^2(dt')^2 - (dx')^2$ as measured by the S' observer. But how can this be? This can be true only if each side of equation 2.9 is equal to a constant. *Thus, the quantity $c^2(dt)^2 - (dx)^2$ is an invariant. That is, it is the same in all inertial systems. This quantity is called the invariant interval and is denoted by $(ds)^2$. Hence the invariant interval is given by*

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

The invariant interval is thus a constant in spacetime. All observers, regardless of their state of motion, agree on this value in spacetime. If the other two space dimensions are included, the invariant interval in four-dimensional spacetime becomes

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

The invariant interval of spacetime is something of a strange quantity to us. In ordinary space, not spacetime, an invariant interval is given by the Pythagorean theorem as

$$(ds)^2 = (dx)^2 + (dy)^2 = (dx')^2 + (dy')^2 \tag{2.12}$$

as shown in figure 2.5, where ds is the invariant, and is seen to be nothing more

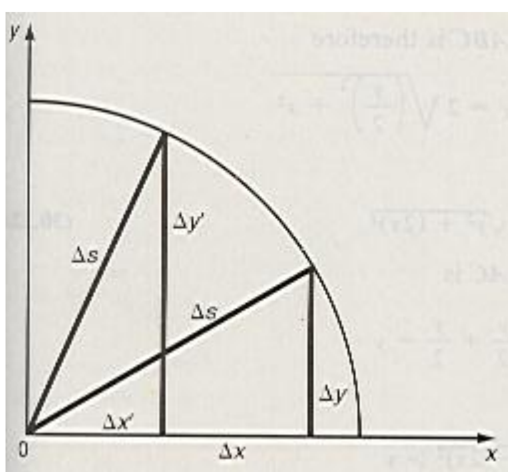

Figure 2.5 The invariant interval of space.

than the radius of the circle shown in figure 2.5 and given by equation 2.12. That is, equation 2.12 is of the form of the equation of a circle $r^2 = x^2 + y^2$. Even though dx

and dx' are different, and dy and dy' are different, the quantity ds is always the same positive quantity.

Now let us look at equation 2.10 for the invariant interval in spacetime. First, however, let $ct = \tau$ as we did previously in equation 2.1. Then we can express the invariant interval, equation 2.10, as

$$(ds)^2 = (d\tau)^2 - (dx)^2 \tag{2.13}$$

Because of the minus sign in front of $(dx)^2$, the equation is not the equation of a circle ($x^2 + y^2 = r^2$), but is rather the equation of a hyperbola, $x^2 - y^2 = constant$.

The interval between two points in Euclidean geometry is represented by the hypotenuse of a right triangle and is given by the Pythagorean theorem: The square of the hypotenuse is equal to the *sum* of the squares of the other two sides of the triangle. However, *the square of the interval ds in spacetime is not equal to the sum of the squares of the other two sides, but to their difference. Thus, the Pythagorean theorem of Euclidean geometry does not hold in spacetime. Therefore, spacetime is not Euclidean. This new type of geometry described by equation 2.13 is sometimes called flat-hyperbolic geometry.* However, since hyperbolic geometry is another name for the non-Euclidean geometry of the Russian mathematician, Nikolai Ivanovich Lobachevski (1793-1856), rather than calling spacetime hyperbolic, we say that spacetime is non-Euclidean. *Space by itself is Euclidean, but spacetime is not. The fact that spacetime is not Euclidean accounts for the apparently strange characteristics of length contraction and time dilation* as we will see shortly. The minus sign in equation 2.13 is the basis for all the differences between space and spacetime.

Also, because of that minus sign in equation 2.13, $(ds)^2$ can be positive, negative, or zero. When $(d\tau)^2 > (dx)^2$, $(ds)^2$ is positive. Because the time term predominates, the world line in spacetime is called timelike and is found in the future light cone. When $(dx)^2 > (d\tau)^2$, $(ds)^2$ is negative. Because the space term predominates in this case, the world line is called spacelike. A spacelike world line lies outside the light cone in the region called elsewhere, figure 2.4. When $(dx)^2 = (d\tau)^2$, $(ds)^2$ is equal to zero. In this case, $(dx) = d\tau = (cdt)$. Hence, $dx = cdt$, or $dx/dt = c$. But dx/dt is a velocity. For it to equal c, it must be the world line of something moving at the speed of light. Thus $(ds)^2 = 0$ represents a light ray and the world line is called lightlike. Lightlike world lines make up the light cone.

Another characteristic of Euclidean space is that the straight line is the shortest distance between two points. Now we will see that *in non-Euclidean spacetime, the straight line is the longest distance between two points.* Consider the distance traveled along the two space paths of figure 2.6(a). The distance traveled along path AB in Euclidean space is found from the Pythagorean Theorem as

$$s_{AB} = \sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

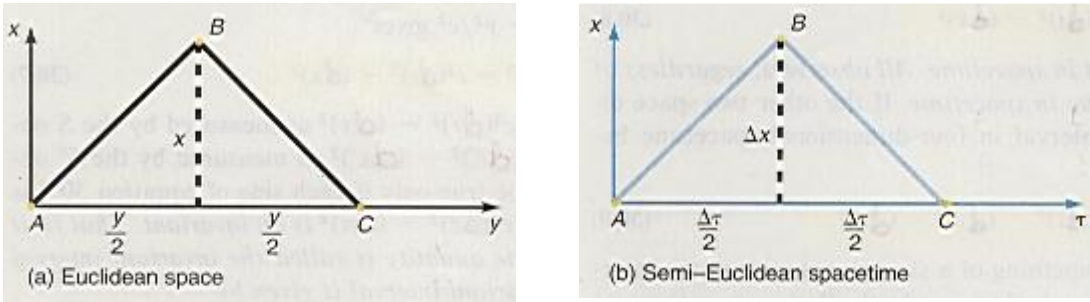

Figure 2.6 Space versus spacetime.

And the distance along path *BC* is similarly

$$s_{BC} = \sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

The total distance traveled along path *ABC* is therefore

$$s_{ABC} = s_{AB} + s_{BC} = 2\sqrt{\left(\frac{y}{2}\right)^2 + x^2}$$

or

$$s_{ABC} = \sqrt{y^2 + (2x)^2} \tag{2.14}$$

The total distance traveled along path *AC* is

$$s_{AC} = \frac{y}{2} + \frac{y}{2} = y$$

But since

$$\sqrt{y^2 + (2x^2)} > y$$

the round-about path *ABC* is longer than the straight line path *AC*, as expected.

Example 2.2

Path length in Euclidean space. If y = 8.00 and x = 3.00 in figure 2.6(a), find the path lengths s_{ABC} and s_{AC}.

Solution

The length of the path along *ABC*, found from equation 2.14, is

$$s_{ABC} = \sqrt{y^2 + (2x^2)} = \sqrt{(8.00)^2 + (2(3.00))^2}$$
$$= 10.0$$

The length of path AC is simply

$$s_{AC} = y = 8.00$$

Thus, the straight line path in space is shorter than the round-about path.

Go to Interactive Example

Let us now look at the same problem in spacetime, as shown in figure 2.6(b). The distance traveled through spacetime along path AB is found by the invariant interval, equation 2.13, as

$$ds_{AB} = \sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$

Whereas the distance traveled through spacetime along path BC is

$$ds_{BC} = \sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$

The total distance traveled through spacetime along path ABC is thus,

$$ds_{ABC} = ds_{AB} + ds_{BC}$$
$$= 2\sqrt{\left(\frac{d\tau}{2}\right)^2 - (dx)^2}$$
$$ds_{ABC} = \sqrt{(d\tau)^2 - (2dx)^2} \tag{2.15}$$

Whereas the distance traveled through spacetime along the path AC is

$$ds_{AC} = \frac{d\tau}{2} + \frac{d\tau}{2} = d\tau$$

But comparing these two paths, ABC and AC, we see that

$$\sqrt{(d\tau)^2 - (2dx)^2} < d\tau \tag{2.16}$$

Therefore, the distance through spacetime along the round-about path ABC is less than the straight line path AC through spacetime. Thus, the shortest distance *between two points in spacetime is not the straight line. In fact the straight line is the longest distance between two points in spacetime.* These apparently strange effects of relativity occur because spacetime is non-Euclidean. (It is that minus sign again!)

Example 2.3

Path length in non-Euclidean spacetime. If $d\tau = 8.00$ and $dx = 3.00$ in figure 2.6(b), find the path lengths ds_{ABC} and ds_{AC}.

Solution

The interval along path ABC, found from equation 2.15, is

$$ds_{ABC} = \sqrt{(d\tau)^2 - (2dx)^2} = \sqrt{(8.00)^2 - (2(3.00))^2}$$
$$= 5.29$$

The interval along path AC is

$$ds_{AC} = d\tau = 8.00$$

Hence,

$$ds_{ABC} < ds_{AC}$$

and the straight line through spacetime is greater than the round-about line through spacetime.

Go to Interactive Example

The straight line AC in spacetime is the world line of an object or clock at rest at the origin of the coordinate system. The spacetime interval for a clock at rest ($dx = 0$) is therefore

$$(ds)^2 = (d\tau)^2 - (dx)^2 = (d\tau_0)^2$$

or

$$ds = d\tau_0 \tag{2.17}$$

The subscript 0 has been used on τ to indicate that this is the time when the clock is at rest. *The time read by a clock at rest is called its proper time. But since this proper time is also equal to the spacetime interval, equation 2.17, and this spacetime interval is an invariant, it follows that the interval measured along any timelike world line is equal to its proper time.* If a clock is carried along with a body from A to B, ds_{AB} is the time that elapses on that clock as it moves from A to B, and ds_{BC} is the time that elapses along path BC. *Hence, from equation 2.16, the time elapsed along path ABC is less than the time elapsed along path AC. Thus, if two clocks started out synchronized at A, they read different times when they come together at* *point C. It is therefore sometimes said that time, like distance, is a route-dependent quantity.* The path *ABC* represents an accelerated path. (Actually the acceleration occurs almost instantaneously at the point *B*.) Hence *the lapse of proper time for an accelerated observer is less than the proper time for an observer at rest. Thus, time must slow down during an acceleration,* a result that we will confirm in our study of general relativity.

In chapter 1 we discussed the twin paradox, whereby one twin became an astronaut and traveled into outer space while the second twin remained home on earth. The Lorentz time dilation equation showed that the traveling astronaut, on his return, would be younger than his stay-at-home twin. Figure 2.6(b) is essentially a spacetime diagram of the twin paradox. The world line through spacetime for the stay-at-home twin is shown as path *AC*, whereas the world line for the astronaut is given by path *ABC*. Path *ABC* through spacetime is curved because the astronaut went through an acceleration phase in order to turn around to return to earth. Hence, the astronaut can no longer be considered as an inertial observer. Since the stay-at-home twin's path *AC* is a straight line in spacetime, she is an inertial observer. As we have just seen in the last paragraph, the time elapsed along path *ABC*, the astronaut's path, is less than the time elapsed along path *AC*, the stay-at-home's path. Thus the astronaut does indeed return home younger than his stay-at-home twin.

Perhaps one of the most important characteristics of the invariant interval is that it allows us to draw a good geometrical picture of spacetime as it is seen by different observers. For example, a portion of spacetime for a stationary observer *S* is shown in figure 2.7. The x and τ coordinates of *S* are shown as the orthogonal axes. The light lines *OL* and *OL'* are drawn at angles of 45^0. The interval, equation 2.13, is drawn for a series of values of x and τ and appear as the family of hyperbolas in the figure. (We might note that if spacetime were Euclidean the intervals would have been a family of concentric circles around the origin *O* instead of these hyperbolas.) The hyperbolas drawn about the τ-axis lie in the light cone future, while the hyperbolas drawn about the x-axis lie elsewhere. The interval has positive values within the light cone and negative values elsewhere.

A frame of reference *S'*, moving at the velocity v, would have for the world line of its origin of coordinates, a straight line through spacetime inclined at an angle θ given by

$$\theta = \tan^{-1}\frac{v}{c} \tag{2.3}$$

For example, if *S'* is moving at a speed of $c/2$, $\theta = 26.6^0$. This world line is drawn in figure 2.7. But the world line of the origin of coordinates ($x' = 0$) is the time axis τ' of the *S'* frame, and is thus so labeled in the diagram. *Where* τ' *intersects the family of hyperbolas at d*s = 1, 2, 3, … , it establishes the time scale along the τ'*-axis as* τ' =

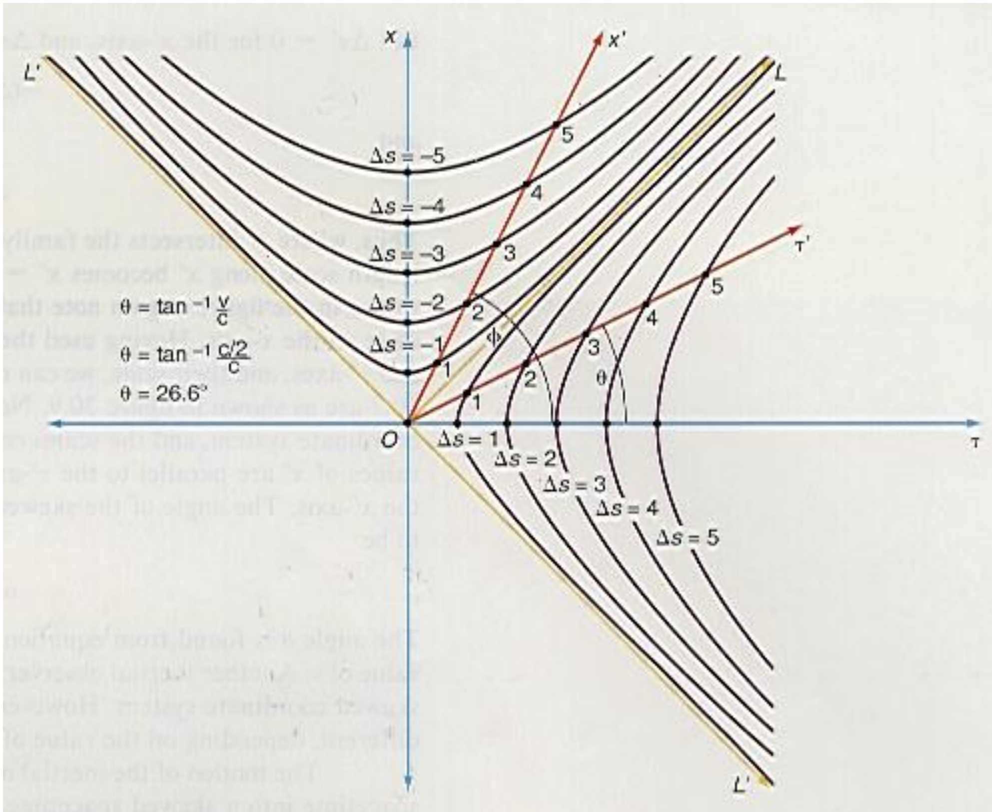

Figure 2.7 The invariant interval on a spacetime map.

1, 2, 3, … (Recall that because $(ds)^2 = (d\tau')^2 - (dx')^2$, and the origin of the coordinate system, $dx' = 0$, hence $ds = d\tau'$.) *Note that the scale on the τ'-axis is not the same as the scale on the τ-axis.*

To draw the x'-axis on this graph, we note that the x'-axis represents all the points for which $t' = 0$. The Lorentz equation for t' was given in chapter 33 by equation 1.50 as

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$$

For $t' = 0$, we must have

$$t = \frac{xv}{c^2}$$

or

$$x = \frac{c^2}{v} t = \frac{c}{v}(ct)$$

$$x = \frac{c}{v}\tau \tag{2.18}$$

Equation 2.18 is the equation of a straight line passing through the origin with the slope c/v. *This line represents the x'-axis because it results from setting t' = 0 in the Lorentz equation.* Because the slope of the τ'-axis was given by $\tan\theta = v/c$, the triangle of figure 2.8 can be drawn. Note that we can write the ratio of c/v, the slope of the x'-axis, as

$$\tan \phi = \frac{c}{v}$$

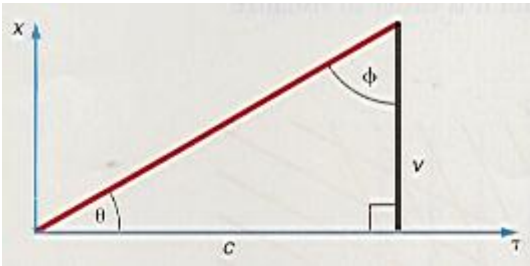

Figure 2.8 Determining the slope of the x'-axis.

But from the figure $\theta + \phi = 90^0$. Hence, the angle for the slope of the x'-axis must be $\phi = 90^0 - \theta$ In our example, $\theta = 26.6^0$, thus $\phi = 63.4^0$. The x'-axis is drawn in figure 2.7 at this angle. Note that the x'-axis makes an angle ϕ with the τ-axis, but an angle θ with the x-axis. The intersection of the x'-axis with the family of hyperbolas establishes the scale for the x'-axis. The interval is

$$(ds)^2 = (d\tau')^2 - (dx')^2$$

But $d\tau' = 0$ for the x'-axis, and ds is a negative quantity elsewhere, hence

$$-(dx')^2 = -(ds)^2$$

and

$$dx' = +\,ds$$

Thus, where x' intersects the family of hyperbolas at $ds = -1, -2, -3, \ldots$ the length scale along x' becomes $x' = 1, 2, 3, \ldots$ The scale on the x'-axis is now shown in the figure. Again note that the scale on the x'-axis is not the same as the scale on the x-axis. Having used the hyperbolas for the interval to establish the x'- and τ'-axes, and their scale, we can now dispense with them and the results of figure 2.7 are as shown in figure 2.9. Notice that the S' frame of reference is a skewed coordinate system, and the scales on S' are not the same as on S. Lines of constant values of x' are parallel to the τ'-axis, whereas lines of constant τ' are parallel to the x'-axis. The angle of the skewed coordinate system α is found from the figure to be

$$\alpha = 90^0 - 2\theta \tag{2.19}$$

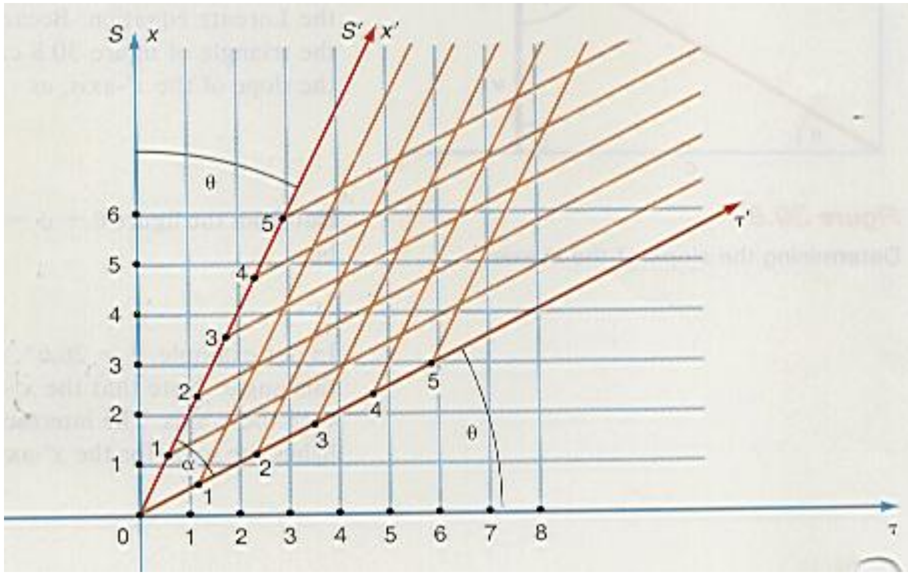

Figure 2.9 Relation of S and S' frame of references.

The angle θ is found from equation 2.3. This *S*' frame is unique for a particular value of *v*. Another inertial observer moving at a different speed would have another skewed coordinate system. However, the angle θ and hence, the angle α, would be different, depending on the value of *v*.

The motion of the inertial observer *S*' seems to warp the simple orthogonal spacetime into a skewed spacetime. The length contraction and time dilation can easily be explained by this skewed spacetime. Figures 2.10 through 2.15 are a series of spacetime diagrams based on the invariant interval, showing length contraction, time dilation, and simultaneity.

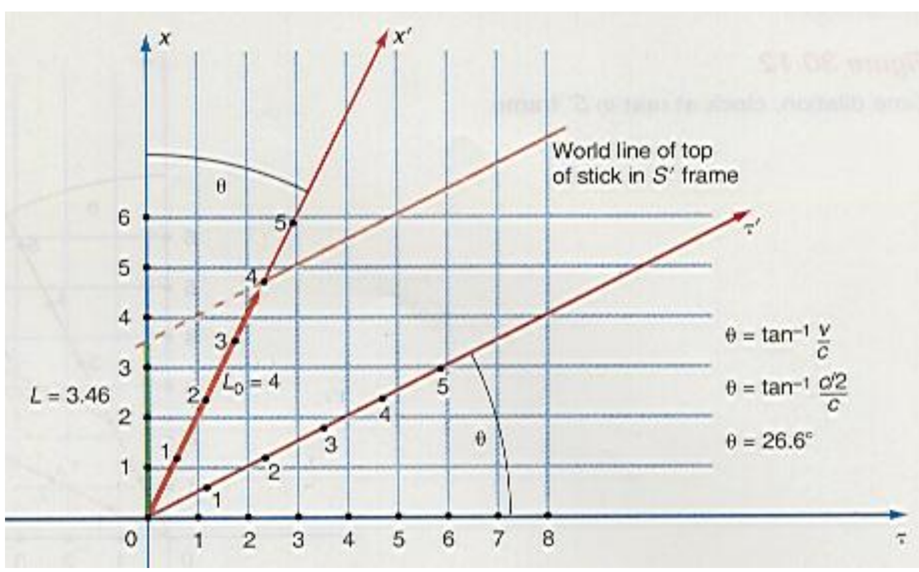

Figure 2.10 Length contraction, rod at rest in S' frame.

Figure 2.10 represents a rod 4.00 units long at rest in a rocket ship S', moving at a speed of $c/2$. The world line of the top of the stick in S' is drawn parallel to the τ'-axis. (Any line parallel to the τ'-axis has one and only one value of x' and thus represents an object at rest in S'.)

If the world line is dashed backward to the x-axis, it intersects the x-axis at $x = 3.46$, which is the length of the rod L, as observed by the S frame observer. Thus, the rod at rest in the moving rocket frame appears contracted to the observer on earth, the S frame. The contraction of the moving rod is, of course, the Lorentz contraction. With the spacetime diagram it is easier to visualize.

Figure 2.11 shows the same Lorentz contraction but as viewed from the S' frame. A rod 4.00 units long L_0 is at rest in the S frame, the earth. An observer in

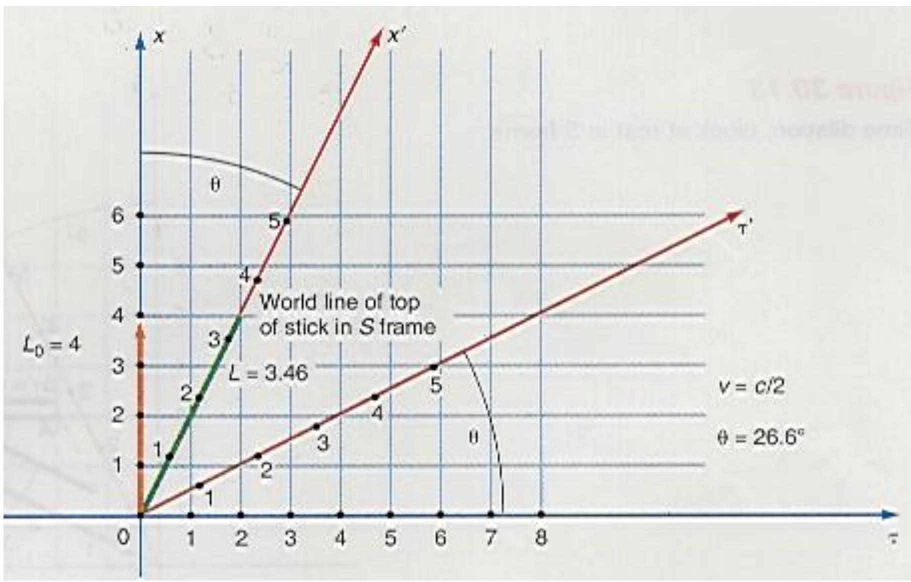

Figure 2.11 Length contraction, rod at rest in S frame.

the rocket ship frame, the S' frame, considers himself to be at rest while the earth is moving away from him at a velocity $-v$. The astronaut sees the world line, which emanates from the top of the rod, as it intersects his coordinate system. The length of the rod that he sees is found by drawing the world line of the top of the rod in the S frame, as shown in the figure. This world line intersects the x'-axis at the position $x' = 3.46$. Hence, the rocket observer measures the rod on earth to be only 3.46 units long, the length L. Thus, the rocket ship observer sees the same length contraction. *The cause of these contractions is the non-Euclidity of spacetime.*

The effect of time dilation is also easily explained by the spacetime diagram, figure 2.12. A clock is at rest in a moving rocket ship at the position $x' = 2$. Its world line is drawn parallel to the τ'-axis, as shown. Between the occurrence of the events A and B a time elapses on the S' clock of $d\tau' = 4.0 - 2.0 = 2.0$, as shown in the figure. This time interval, when observed by the S frame of the earthman, is found by dropping the dashed lines from the events A and B down to the τ-axis. (These lines are parallel to the x-axis, but because S is an orthogonal frame, they are also perpendicular to the τ-axis.) The time interval elapsed on earth is read from the graph as $d\tau = 5.9 - 3.6 = 2.3$. A time lapse of 2 s on the rocket ship clock would

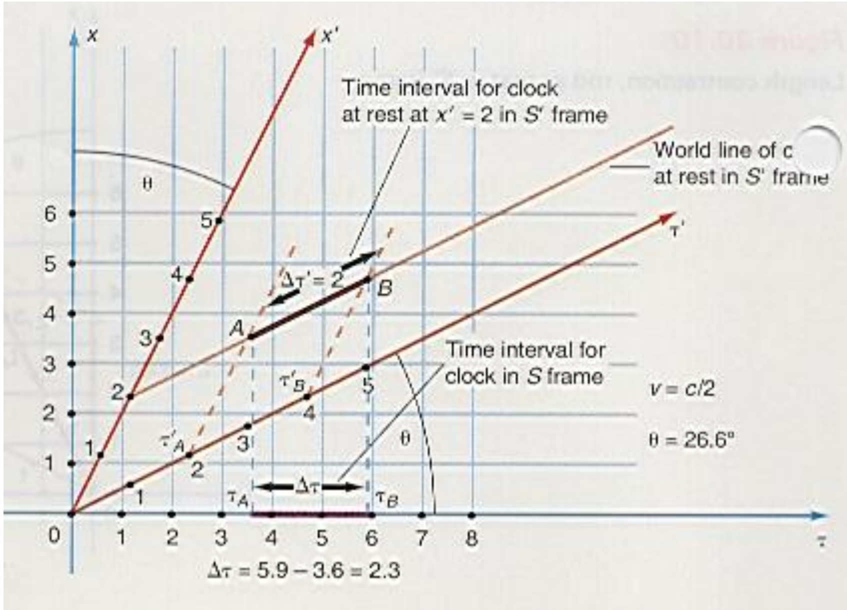

Figure 2.12 Time dilation, clock at rest in S′ frame.

appear as a lapse of 2.3 s on earth. Thus the moving clock in *S*' is running at a slower rate than a clock in *S*. Time has slowed down in the moving rocket ship. This is, of course, the Lorentz time dilation effect.

The inverse problem of time dilation is shown in figure 2.13. Here a clock is at rest on the earth, the *S* frame, at the position $x = 3$. The world line of the clock is

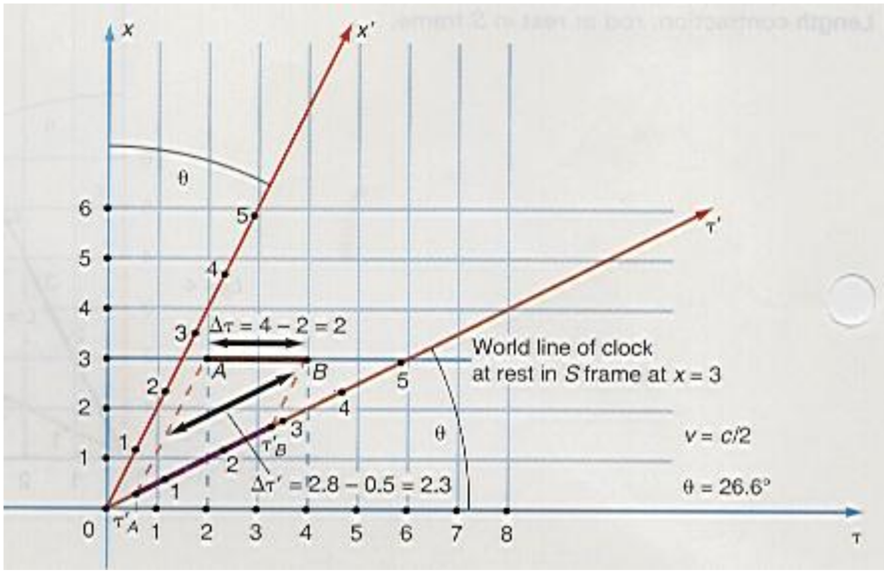

Figure 2.13 Time dilation, clock at rest in S frame.

drawn parallel to the τ-axis. The occurrence of two events, *A* and *B*, are noted and the time interval elapsed between these two events on earth is $d\tau = 4.0 - 2.0 = 2.0$. The same events *A* and *B* are observed in the rocket ship, and the time of these events as observed on the rocket ship is found by drawing the dashed lines parallel to the x'-axis to where they intersect the τ'-axis. Thus, event *A* occurs at $\tau'_A = 0.5$, and event *B* occurs at $\tau'_B = 2.8$. The elapsed time on the rocket ship is thus

$$d\tau' = \tau_B' - \tau_A' = 2.8 - 0.5 = 2.3$$

From the point of view of the rocket observer, he is at rest, and the earth is moving away from him at a velocity $-v$. Hence, he sees an elapsed time on the moving earth of 2 s while his own clock records a time interval of 2.3 s. He therefore concludes that time has slowed down on the moving earth.

Another explanation for this time dilation can be found in the concept of *simultaneity*. If we look back at figure 2.12 we see that the same event *A* occurs at the times $\tau_A = 3.6$ and $\tau'_A = 2.0$, whereas event *B* occurs at the times $\tau_B = 5.9$ and $\tau'_B = 4$. *The same event does not occur at the same time in the different coordinate systems.* Because the events occur at different times their time intervals should be expected to be different also. In fact, a more detailed picture of simultaneity can be found in figures 2.14 and 2.15.

Figure 2.14 shows two events *A* and *B* that occur simultaneously at the time $\tau' = 2$ on the moving rocket ship. However, the earth observer sees the two events

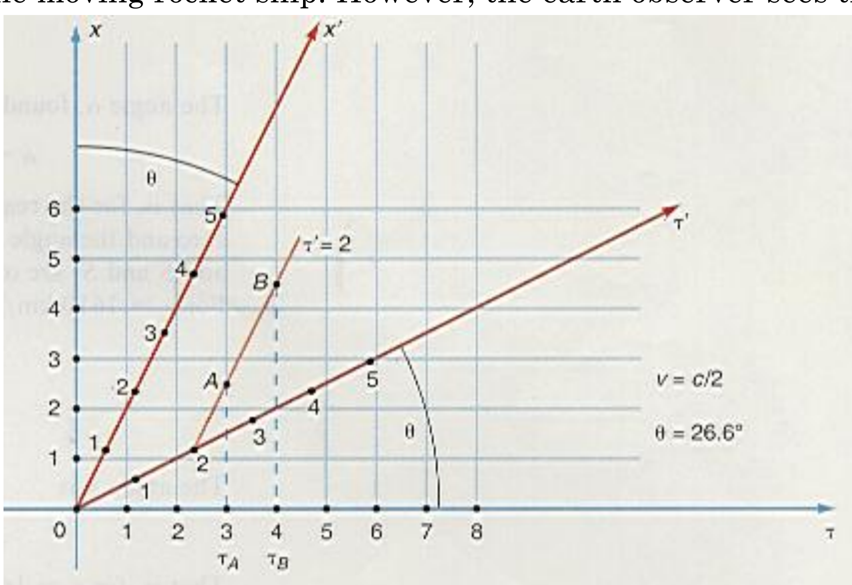

Figure 2.14 Simultaneity, two events simultaneous in S' frame.

occurring, not simultaneously, but rather at the two times $\tau_A = 3$ and $\tau_B = 4$. That is, the earth observer sees event *A* happen before event *B*. This same type of effect is shown in figure 2.15, where the two events *A* and *B* now occur simultaneously at $\tau = 4$ for the earth observer. However the rocket ship observer sees event *B* occurring at

$\tau'_B = 1.6$ and event A at $\tau'_A = 2.7$. Thus, to the rocket ship observer events A and B are not simultaneous, but rather event B occurs before event A.

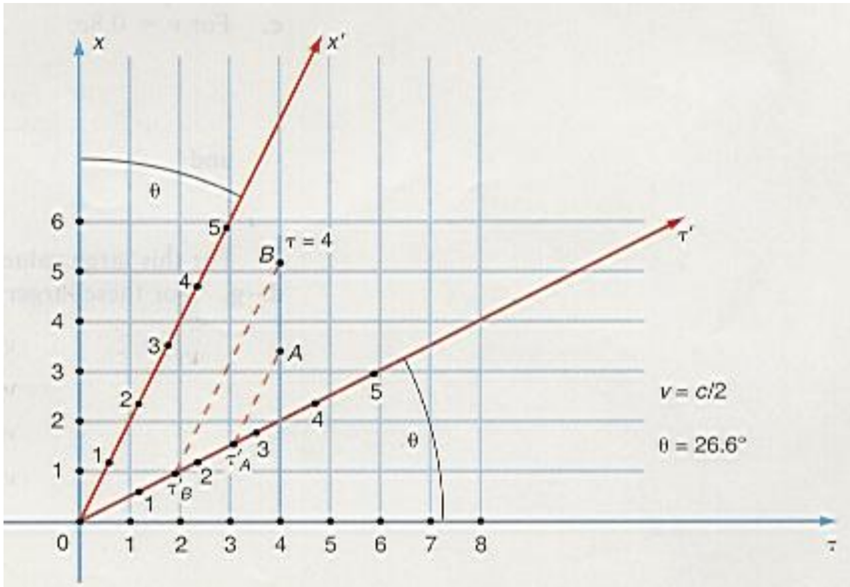

Figure 2.15 Simultaneity, two events simultaneous in S frame.

In summary, these spacetime diagrams are based on the invariant interval. Because the invariant interval is based on hyperbolas, spacetime is non-Euclidean. The S' frame of reference becomes a skewed coordinate system and the scales of the S' frame are not the same as the scales on the S frame.

Example 2.4

The skewing of the spacetime diagram with speed. Find the angles θ and α for a spacetime diagram if (a) v = 1610 km/hr = 1000 mph = 477 m/s, (b) v = 1610 km/s = 1000 miles/s, (c) $v = 0.8c$, (d) $v = 0.9c$, (e) $v = 0.99c$, (f) $v = 0.999c$, and (g) $v = c$.

Solution

a. The angle θ of the spacetime diagram, found from equation 2.3, is

$$\theta = \tan^{-1}\frac{v}{c}$$
$$= \tan^{-1}\frac{477 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}$$
$$= (8.54 \times 10^{-5})^0$$

The angle α, found from equation 2.19, is

$$\alpha = 90^0 - 2\theta = 90^0 - 2(8.54 \times 10^{-5})^0 = 90^0$$

That is, for the reasonably large speed of 1000 mph, the angle θ is effectively zero and the angle $\alpha = 90^0$. There is no skewing of the coordinate system and S and S' are orthogonal coordinate systems.

b. For v = 1610 km/s, the angle θ is

$$\theta = \tan^{-1}\frac{v}{c} = \tan^{-1}\frac{1.61 \times 10^6 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}}$$
$$= 0.31^0$$

The angle α is

$$\alpha = 90^0 - 2\theta = 90^0 - 2(0.31^0) = 89.4^0$$

That is, for v = 1610 km/s = 3,600,000 mph, the τ'- and x'-axes are just barely skewed.

c. For $v = 0.8c$,

$$\theta = \tan^{-1}\frac{v}{c} = \tan^{-1}\frac{0.8c}{c} = 38.7^0$$

and

$$\alpha = 90^0 - 2\theta = 90^0 - 2(38.7) = 12.6^0$$

For this large value of v, the axes are even more skewed than in figure 2.8.

d.-g. For these larger values of v, equations 2.3 and 2.19 give

$v = 0.9c$;	$\theta = 41.9^0$;	$\alpha = 6.2^0$
$v = 0.99c$;	$\theta = 44.7^0$;	$\alpha = 0.576^0$
$v = 0.999c$;	$\theta = 44.97^0$;	$\alpha = 0.057^0$
$v = c$;	$\theta = 45^0$;	$\alpha = 0^0$

Hence, as v gets larger and larger the angle θ between the coordinate axes becomes larger and larger, eventually approaching 45^0. The angle α gets smaller until at $v = c$, α has been reduced to zero and the entire S' frame of reference has been reduced to a line.

Go to Interactive Example

2.3 The General Theory of Relativity

We saw in the special theory of relativity that the laws of physics must be the same in all inertial reference systems. *But what is so special about an inertial reference* *system? The inertial reference frames are, in a sense, playing the same role as Newton's absolute space.* That is, absolute space has been abolished only to replace it by absolute inertial reference frames. Shouldn't the laws of physics be the same in all coordinate systems, whether inertial or noninertial? The inertial frame should not be such a privileged frame. But clearly, accelerations can be easily detected, whereas constant velocities cannot. How can this very obvious difference be reconciled? That is, we must show that even all accelerated motions are relative. How can this be done?

Let us consider the very simple case of a mass m on the floor of a rocket ship that is at rest in a uniform gravitational field on the surface of the earth, as depicted in figure 2.16(a). The force acting on the mass is its weight w, which we write as

$$F = w = mg \tag{2.20}$$

Let us now consider the case of the same rocket ship in interstellar space far removed from all gravitational fields. Let the rocket ship now accelerate upward, as in figure 2.16(b), with an acceleration a that is numerically equal to the acceleration due to gravity g, that is, $a = g = 9.80\ \text{m/s}^2$. The mass m that is sitting on the floor of the rocket now experiences the force, given by Newton's second law as

$$F = ma = mg = w \tag{2.21}$$

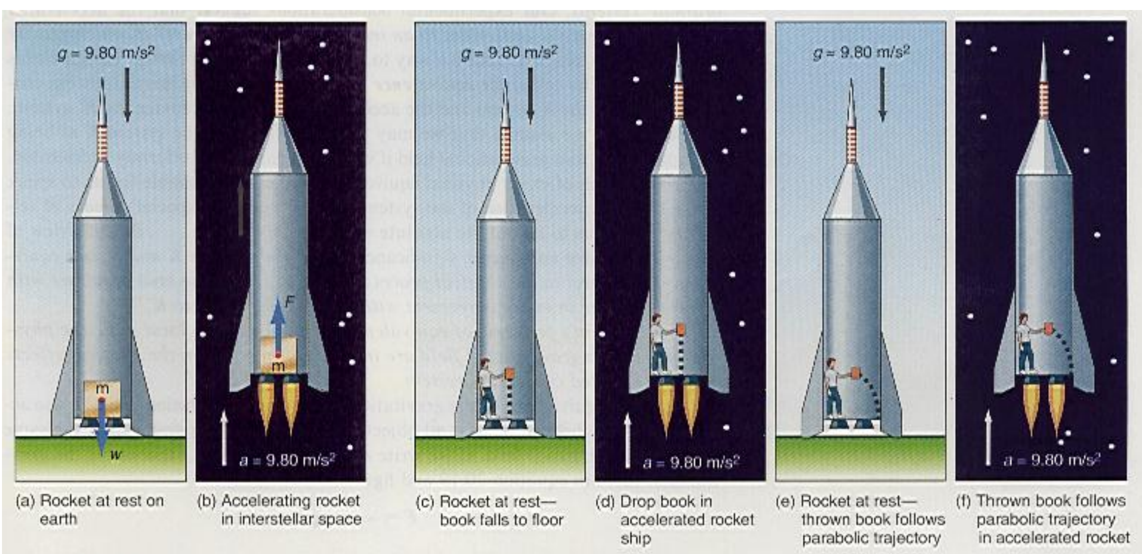

Figure 2.16 An accelerated frame of reference is equivalent to an inertial frame of reference plus gravity.

That is, the mass m sitting on the floor of the accelerated rocket experiences the same force as the mass m sitting on the floor of the rocket ship when it is at rest in the uniform gravitational field of the earth. Therefore, there seems to be some relation between accelerations and gravity.

Let us experiment a little further in the rocket ship at rest by holding a book out in front of us and then dropping it, as in figure 2.16(c). The book falls to the floor and if we measured the acceleration we would, of course, find it to be the acceleration due to gravity, g = 9.80 m/s^2. Now let us take the same book in the accelerated rocket ship and again drop it, as in figure 2.16(d). An inertial observer outside the rocket would see the book stay in one place but would see the floor accelerating upward toward the book at the rate of a = 9.80 m/s^2. The astronaut in the accelerated rocket ship sees the book fall to the floor with the acceleration of 9.80 m/s^2 just as the astronaut at rest on the earth observed.

The astronaut in the rocket at rest on the earth now throws the book across the room of the rocket ship. He observes that the book follows the familiar parabolic trajectory of the projectile and that is again shown in figure 2.16(e). Similarly, the astronaut in the accelerated rocket also throws the book across the room. An outside inertial observer would observe the book moving across the room in a straight line and would also see the floor accelerating upward toward the book. The accelerated astronaut would simply see the book following the familiar parabolic trajectory it followed on earth, figure 2.16(f).

Hence, the same results are obtained in the accelerated rocket ship as are found in the rocket ship at rest in the gravitational field of the earth. Thus, *the effects of gravity can be either created or eliminated by the proper choice of coordinate systems.* Our experimental considerations suggest that *the accelerated frame of reference is equivalent to an inertial frame of reference in which gravity is present.* Einstein, thus found a way to make accelerations relative. He stated his results in what he called the ***equivalence principle.*** Calling the inertial system containing gravity the K system and the accelerated frame of reference the K' system, Einstein said, “we assume that we may just as well regard the system K as being in a space free from gravitational field if we then regard K as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system, just as the usual (special) theory of relativity forbids us to talk of the absolute velocity of a system… But this view of ours will not have any deeper significance unless *the systems K and K' are equivalent with respect to all physical processes, that is, unless the laws of nature with respect to K are in entire agreement with those with respect to K'*”[2]

Einstein’s principle of equivalence is stated as: on a local scale the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system.

The equivalence of the gravitational field and acceleration “fields” also accounts for the observation that all objects, regardless of their size, fall at the same rate in a gravitational field. If we write $m_{\mathbf{g}}$ for the mass that experiences the gravitational force in equation 2.20 and figure 2.16(a), then

$$F = w = m_{\mathbf{g}}g$$

2[2]“On the Influence of Gravitation on the Propagation of Light,” from A. Einstein, *Annalen der Physik* 35, 1911, in *The Principle of Relativity,* Dover Publishing Co.

And if we write m_i for the inertial mass that resists the motion of the rocket in figure 2.16(b) and equation 2.21, then

$$F = m_i a = m_i g$$

Since we have already seen that the two forces are equal, by the equivalence principle, it follows that

$$m_g = m_i$$

That is, the gravitational mass is in fact equal to the inertial mass. Thus, the equivalence principle implies the equality of inertial and gravitational mass and this is the reason why all objects of any size fall at the same rate in a gravitational field.

As a final example of the equivalence of a gravitational field and an acceleration let us consider an observer in a closed room, such as a nonrotating space station in interstellar space, far removed from all gravitating matter. This space station is truly an inertial coordinate system. Let the observer place a book in front of him and then release it, as shown in figure 2.17(a). Since there are no forces

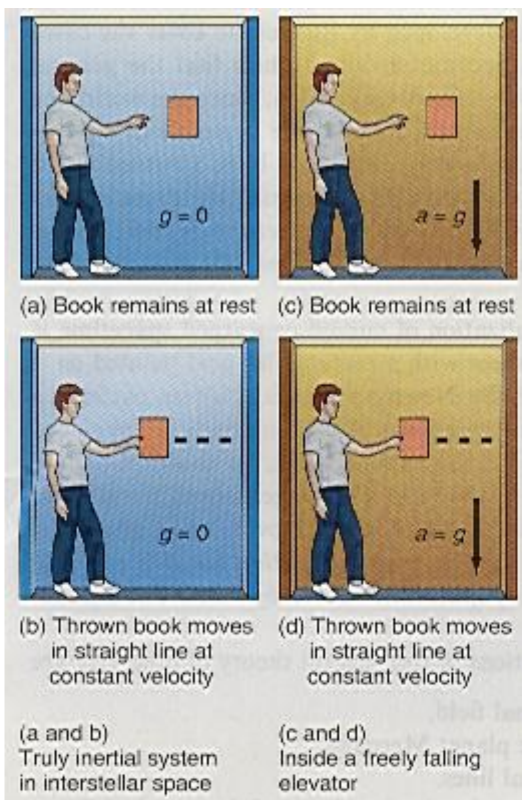

Figure 2.17 A freely falling frame of reference is locally the same as an inertial frame of reference.

present, not even gravity, the book stays suspended in space, at rest, exactly where the observer placed it. If the observer then took the book and threw it across the room, he would observe the book moving in a straight line at constant velocity, as shown in figure 2.17(b).

Let us now consider an elevator on earth where the supporting cables have broken and the elevator goes into free-fall. An observer inside the freely falling elevator places a book in front of himself and then releases it. The book appears to that freely falling observer to be at rest exactly where the observer placed it, figure 2.17(c). (Of course, an observer outside the freely falling elevator would observe both the man and the book in free-fall but with no relative motion with respect to each other.) If the freely falling observer now takes the book and throws it across the elevator room he would observe that the book travels in a straight line at constant velocity, figure 2.17(d).

Because an inertial frame is defined by Newton's first law as a frame in which a body at rest, remains at rest, and a body in motion at some constant velocity continues in motion at that same constant velocity, we must conclude from the illustration of figure 2.17 that the freely falling frame of reference acts exactly as an inertial coordinate system to anyone inside of it. *Thus, the acceleration due to gravity has been transformed away by accelerating the coordinate system by the same amount as the acceleration due to gravity.* If the elevator were completely closed, the observer could not tell whether he was in a freely falling elevator or in a space station in interstellar space.

The equivalence principle allows us to treat an accelerated frame of reference as equivalent to an inertial frame of reference with gravity present, figure 2.16, or to consider an inertial frame as equivalent to an accelerated frame in which gravity is absent, figure 2.17. By placing all frames of reference on the same footing, Einstein was then able to *postulate* ***the general theory of relativity, namely, the laws of physics are the same in all frames of reference.***

A complete analysis of the general theory of relativity requires the use of very advanced mathematics, called tensor analysis. However, many of the results of the general theory can be explained in terms of the equivalence principle, and this is the path that we will follow in the rest of this chapter.

From his general theory of relativity, Einstein was quick to see its relation to gravitation when he said, "It will be seen from these reflections that in pursuing the General Theory of Relativity we shall be led to a theory of gravitation, since we are able to produce a gravitational field merely by changing the system of coordinates. *It will also be obvious that the principle of the constancy of the velocity of light in vacuo must be modified."*[3]

Although the general theory was developed by Einstein to cover the cases of accelerated reference frames, it soon became obvious to him that the general theory had something quite significant to say about gravitation. Since the world line of an accelerated particle in spacetime is curved, then by the principle of equivalence, a particle moving under the effect of gravity must also have a curved world line in spacetime. *Hence, the mass that is responsible for causing the gravitational field,*

3[3]"The Foundation of the General Theory of Relativity" from A. Einstein, *Annalen der Physik* 49, 1916 in *The Principle of Relativity,* Dover Publishing Co.

must warp spacetime to make the world lines of spacetime curved. This is sometimes expressed as, matter warps spacetime and spacetime tells matter how to move.

A familiar example of the visualization of curved or **warped spacetime** is the rubber sheet analogy. A flat rubber sheet with a rectangular grid painted on it is stretched, as shown in figure 2.18(a). By Newton's first law, a free particle, a small rolling ball m moves in a straight line as shown. A bowling ball is then placed on the rubber sheet distorting or warping the rubber sheet, as shown in figure 2.18(b). When the small ball m is rolled on the sheet it no longer moves in a straight line path but it now curves around the bowling ball M, as shown. *Thus,*

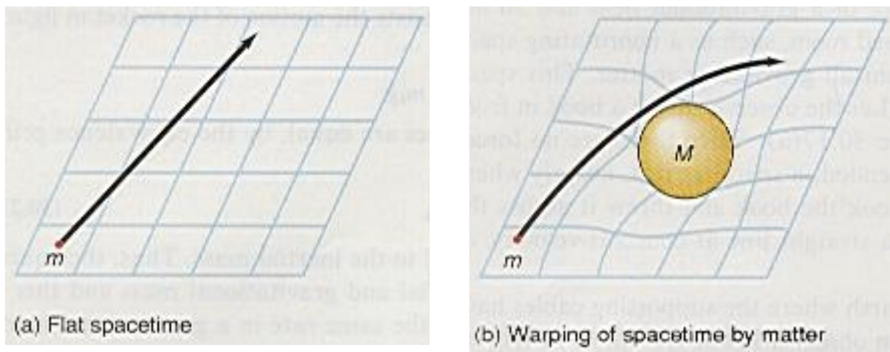

Figure 2.18 Flat and curved spacetime

gravity is no longer to be thought of as a force in the Newtonian tradition but it is rather a consequence of the warping or curvature of spacetime caused by mass. The amount of warping is a function of the mass.

The four experimental confirmations of the general theory of relativity are

1. The bending of light in a gravitational field.
2. The advance of the perihelion of the planet Mercury.
3. The gravitational red shift of spectral lines.
4. The Shapiro experiment, which shows the slowing down of the speed of light near a large mass.

Let us now look at each of these confirmations.

2.4 The Bending of Light in a Gravitational Field

Let us consider a ray of light that shines through a window in an elevator at rest, as shown in figure 2.19(a). The ray of light follows a straight line path and hits the opposite wall of the elevator at the point P. Let us now repeat the experiment, but let the elevator accelerate upward very rapidly, as shown in figure 2.19(b). The ray of light enters the window as before, but before it can cross the room to the opposite wall the elevator is displaced upward because of the acceleration. Instead of the ray of light hitting the wall at the point P, it hits at some lower point Q because of the upward acceleration of the elevator. To an observer in the elevator, the ray of light follows the parabolic path, as shown in figure 2.19(c). Thus, *in the accelerated*

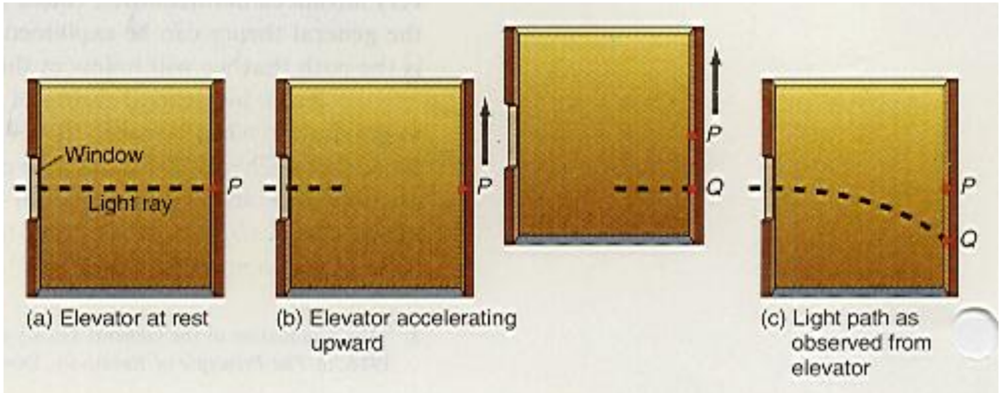

Figure 2.19 The bending of light in an accelerated elevator.

coordinate system of the elevator, light does not travel in a straight line, but instead follows a curved path. But by the principle of equivalence the accelerated elevator can be replaced by a gravitational field. Therefore light should be bent from a straight line path in the presence of a gravitational field. The gravitational field of the earth is relatively small and the bending cannot be measured on earth. However, the gravitational field of the sun is much larger and Einstein predicted in 1916 that rays of light that pass close to the sun should be bent by the gravitational field of the sun.

Another way of considering this bending of light is to say that light has energy and energy can be equated to mass, thus the light-mass should be attracted to the sun. Finally, we can think of this bending of light in terms of the curvature of spacetime caused by the mass of the sun. Light follows the shortest path, called a *geodesic,* and is thus bent by the curvature of spacetime.

Regardless of which conceptual picture we pick, Einstein predicted that a ray of light should be deflected by the sun by the angle of 1.75 seconds of arc. In order to observe this deflection it was necessary to measure the angular deviation between two stars when they are far removed from the sun, and then measure the deflection again when they are close to the sun (see figure 2.20). Of course when they are close to the sun, there is too much light from the sun to be able to see the stars. Hence, to test out Einstein's prediction it was necessary to measure the separation during a total eclipse of the sun. Sir Arthur Eddington led an expedition to the west coast of Africa for the solar eclipse of May 29, 1919, and measured the deflection. On November 6, 1919, the confirmation of Einstein's prediction of the bending of light was announced to the world.

More modern techniques used today measure radio waves from the two quasars, 3c273 and 3c279 in the constellation of Virgo. A quasar is a quasi-stellar object, a star that emits very large quantities of radio waves. Because the sun is very dim in the emission of radio waves, radio astronomers do not have to wait for an eclipse to measure the angular separation but can measure it at any time. On October 8, 1972, when the quasars were close to the sun, radio astronomers measured the angular separation between 3c273 and 3c279 in radio waves and found that the change in the angular separation caused by the bending of the radio waves around the sun was 1.73 seconds of arc, in agreement with the general theory of relativity.

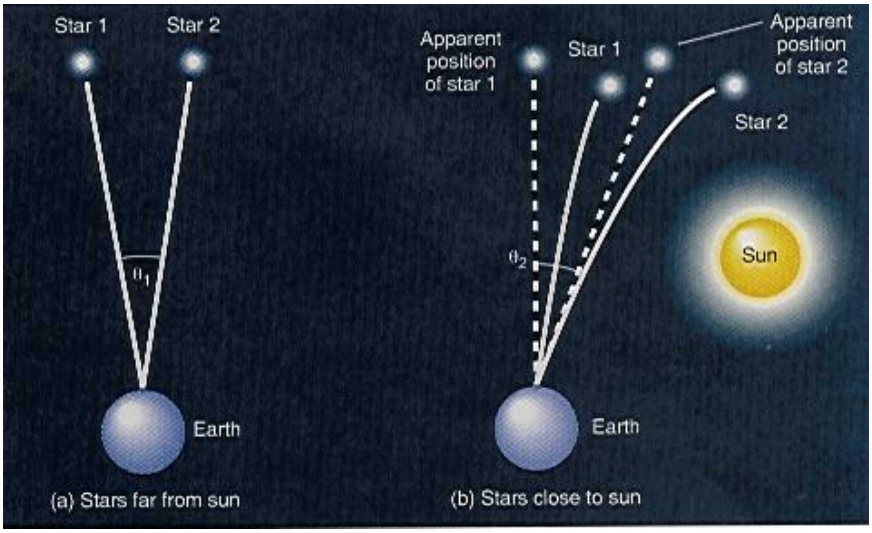

Figure 9.3 Bending of light by the Sun.

2.5 The Advance of the Perihelion of the Planet Mercury

According to Newton's laws of motion and his law of universal gravitation, each planet revolves around the sun in an elliptic orbit, as shown in figure 2.21. The

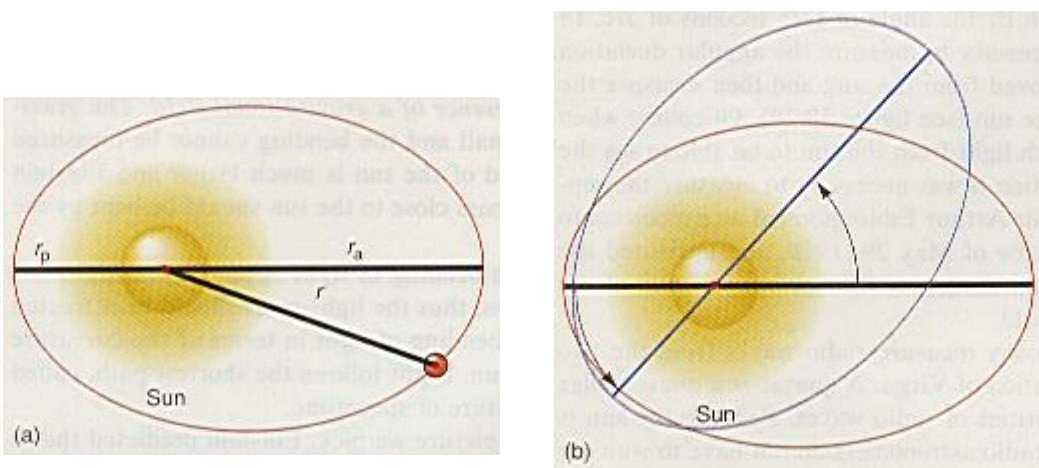

Figure 2.21 Advance of the perihelion of the planet Mercury.

closest approach of the planet to the sun is called its *perihelion distance* r_p, whereas its furthest distance is called its *aphelion distance* r_a. If there were only one planet in the solar system, the elliptical orbit would stay exactly as it is in figure 2.21(a). However, there are other planets in the solar system and each of these planets exerts forces on every other planet. Because the masses of each of these planets is small compared to the mass of the sun, their gravitational effects are also relatively small. These extra gravitational forces cause a perturbation of the elliptical orbit. In particular, they cause the elliptical orbit to rotate in its plane, as shown in figure 2.21(b). The total precession of the perihelion of the planet Mercury is 574 seconds of arc in a century. The perturbation of all the other planets can only explain 531 seconds of arc by the Newtonian theory of gravitation, leaving a discrepancy of 43 seconds of arc per century of the advance of the perihelion of Mercury. Einstein, using the full power of his tensor equations, predicted an advance of the perihelion by 43 seconds of arc per century in agreement with the known observational discrepancy.

2.6 The Gravitational Red Shift

Let us consider the two clocks A and B located at the top and bottom of the rocket, respectively, in figure 2.22(a). The rocket is in interstellar space where we assume

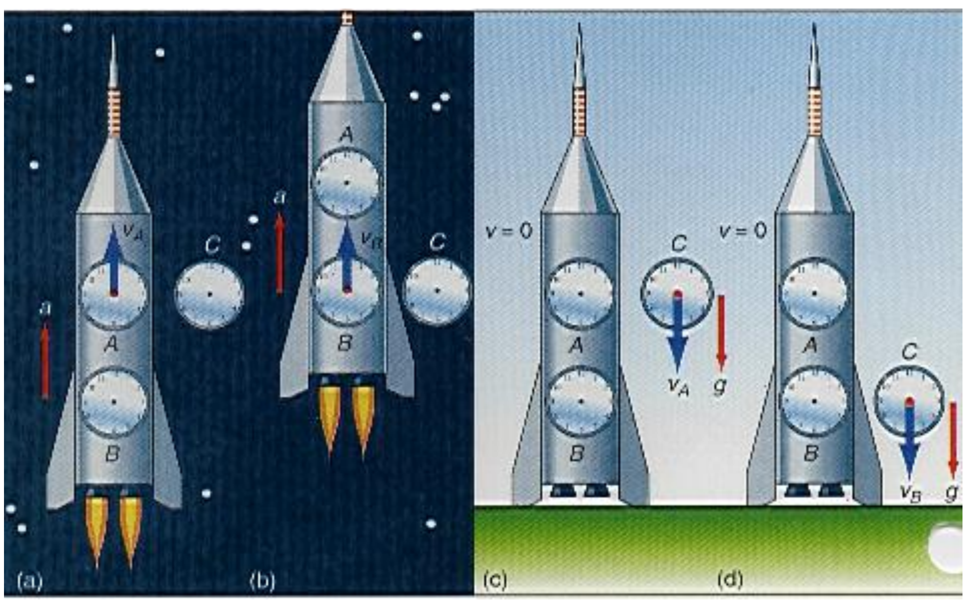

Figure 2.22 A clock in a gravitational field.

that all gravitational fields, if any, are effectively zero. The rocket is accelerating uniformly, as shown. Located in this interstellar space is a clock C, which is at rest. At the instant that the top of the rocket accelerates past clock C, clock A passes clock C at the speed v_A. Clock A, the moving clock, when observed from clock C, the stationary clock, shows an elapsed time Δt_A, given by the time dilation equation 1.64 as

$$\Delta t_C = \frac{\Delta t_A}{\sqrt{1 - v_A^2 / c^2}} \tag{2.22}$$

And since $\sqrt{1-v_A^2/c^2}$ is less than 1, then $\Delta t_C > \Delta t_A$, and the moving clock A runs slow compared to the stationary clock C.

A few moments later, clock B passes clock C at the speed v_B, as in figure 2.22(b). The speed v_B is greater than v_A because of the acceleration of the rocket. Let us read the same time interval Δt_C on clock C when clock B passes as we did for clock A so the two clocks can be compared. The difference in the time interval between the two clocks, B and C, is again given by the time dilation equation 1.64 as

$$\Delta t_C = \frac{\Delta t_B}{\sqrt{1-v_B^2/c^2}} \tag{2.23}$$

Because the time interval Δt_C was set up to be the same in both equations 2.22 and 2.23, the two equations can be equated to give a relation between clocks A and B. Thus,

$$\frac{\Delta t_A}{\sqrt{1-v_A^2/c^2}} = \frac{\Delta t_B}{\sqrt{1-v_B^2/c^2}}$$

Rearranging terms, we get

$$\frac{\Delta t_A}{\Delta t_B} = \frac{\left(1-v_A^2/c^2\right)^{1/2}}{\left(1-v_B^2/c^2\right)^{1/2}}$$

$$\frac{\Delta t_A}{\Delta t_B} = \left(1-v_A^2/c^2\right)^{1/2}\left(1-v_B^2/c^2\right)^{-1/2} \tag{2.24}$$

But the two terms on the right-hand side of equation 2.24 can be expanded by the binomial theorem, equation 1.33, as

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

This is a valid series expansion for $(1-x)^n$ as long as x is less than 1. In this particular case,

$$x = v^2/c^2$$

which is much less than 1, and therefore.

$$(1-x)^n = 1 - nx$$

$$\left(1-v_A^2/c^2\right)^{1/2} = 1-\left(\frac{1}{2}\right)\frac{v_A^2}{c^2} = 1-\frac{v_A^2}{2c^2}$$

and

$$\left(1-v_B^2/c^2\right)^{-1/2} = 1-\left(\frac{-1}{2}\right)\frac{v_B^2}{c^2} = 1+\frac{v_B^2}{2c^2}$$

where again the assumption is made that v is small enough compared to c, to allow us to neglect the terms x^2 and higher in the expansion. Thus, equation 2.24 becomes

$$\frac{\Delta t_A}{\Delta t_B} = \left(1 - \frac{v_A^2}{2c^2}\right)\left(1 + \frac{v_B^2}{2c^2}\right)$$
$$= 1 + \frac{v_B^2}{2c^2} - \frac{v_A^2}{2c^2} - \frac{1}{4}\frac{v_B^2 v_A^2}{c^4}$$

The last term is set equal to zero on the same assumption that the speeds v are much less than c. Finally, rearranging terms,

$$\frac{\Delta t_A}{\Delta t_B} = 1 + \left(\frac{v_B^2}{2} - \frac{v_A^2}{2}\right)\frac{1}{c^2} \tag{2.25}$$

But by Einstein's principle of equivalence, we can equally well say that the rocket is at rest in the gravitational field of the earth, whereas the clock C is accelerating toward the earth in free-fall. When the clock C passes clock A it has the instantaneous velocity v_A, figure 2.22(c), and when it passes clock B it has the instantaneous velocity v_B, figure 2.22(b). We can obtain the velocities v_A and v_B by the law of conservation of energy, that is,

$$\frac{1}{2} mv^2 + \text{PE} = E_0 = \text{Constant} = \text{Total energy} \tag{2.26}$$

The total energy per unit mass, found by dividing equation 2.26 by m, is

$$\frac{v^2}{2} + \frac{\text{PE}}{m} = \frac{E_0}{m}$$

The conservation of energy per unit mass when clock C is next to clock A, obtained with the aid of figure 2.23, is

$$\frac{v_A^2}{2} + \frac{mgh_A}{m} = \frac{E_0}{m}$$

or

$$\frac{v_A^2}{2} + gh_A = \frac{E_0}{m} \tag{2.27}$$

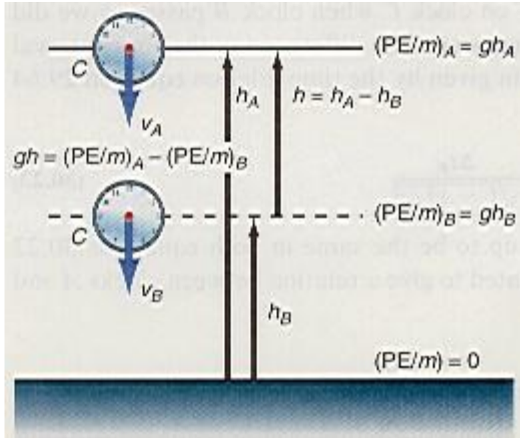

Figure 2.23 Freely falling clock C.

Similarly, when the clock C is next to clock B, the conservation of energy per unit mass becomes

$$\frac{v_B^2}{2} + gh_B = \frac{E_0}{m} \tag{2.28}$$

Subtracting equation 2.27 from equation 2.28, gives

$$\frac{v_B^2}{2} + gh_B - \frac{v_A^2}{2} - gh_A = \frac{E_0}{m} - \frac{E_0}{m} = 0$$

Hence,

$$\frac{v_B^2}{2} - \frac{v_A^2}{2} = gh_A - gh_B = gh \tag{2.29}$$

where h is the distance between A and B, and gh is the gravitational potential energy per unit mass, which is sometimes called the *gravitational potential.* Substituting equation 2.29 back into equation 2.25, gives

$$\frac{\Delta t_A}{\Delta t_B} = 1 + \frac{gh}{c^2} \tag{2.30}$$

For a clearer interpretation of equation 2.30, let us change the notation slightly. Because clock B is closer to the surface of the earth where there is a stronger gravitational field than there is at a height h above the surface where the gravitational field is weaker, we will let

$$\Delta t_B = \Delta t_g$$

and

$$\Delta t_A = \Delta t_f$$

where Δt_g is the elapsed time on a clock in a strong gravitational field and Δt_f is the elapsed time on a clock in a weaker gravitational field. If we are far enough away from the gravitational mass, we can say that Δt_f is the elapsed time in a gravitational-field-**f**ree space. With this new notation equation 2.30 becomes

$$\frac{\Delta t_f}{\Delta t_g} = 1 + \frac{gh}{c^2} \tag{2.31}$$

or

$$\Delta t_f = \Delta t_g \left(1 + \frac{gh}{c^2}\right) \tag{2.31}$$

Since $(1 + gh/c^2) > 0$, *the elapsed time on the clock in the gravitational-field-free space* Δt_f *is greater than the elapsed time on a clock in a gravitational field* Δt_g*. Thus, the time elapsed on a clock in a gravitational field is less than the time elapsed on a clock in a gravity-free space. Hence, a clock in a gravitational field runs slower than a clock in a field-free space.*

We can find the effect of the slowing down of a clock in a gravitational field by placing an excited atom in a gravitational field, and then observing a spectral line from that atom far away from the gravitational field. The speed of the light from that spectral line is, of course, given by

$$c = \lambda\nu = \frac{\lambda}{T} \tag{2.32}$$

where λ is the wavelength of the spectral line, ν is its frequency, and T is the period or time interval associated with that frequency. Hence, if the time interval $\Delta t = T$ changes, then the wavelength of that light must also change. Solving for the period or time interval from equation 2.32, we get

$$T = \frac{\lambda}{c} \tag{2.33}$$

Substituting T from 2.33 for Δt in equation 2.31, we get

$$T_f = T_g \left(1 + \frac{gh}{c^2}\right) \tag{2.34}$$

$$\frac{\lambda_f}{c} = \frac{\lambda_g}{c}\left(1 + \frac{gh}{c^2}\right)$$

$$\lambda_f = \lambda_g \left(1 + \frac{gh}{c^2}\right) \tag{2.35}$$

where λ_g is the wavelength of the emitted spectral line in the gravitational field and λ_f is the wavelength of the observed spectral line in gravity-free space, or at least farther from where the atom is located in the gravitational field. Because the term $(1 + gh/c^2)$ is a positive number, it follows that

$$\lambda_f > \lambda_g \tag{2.36}$$

That is, *the wavelength observed in the gravity-free space is greater than the wavelength emitted from the atom in the gravitational field.* Recall that the visible portion of the electromagnetic spectrum runs from violet light at around 380.0 nm to red light at 720.0 nm. Thus, red light is associated with longer wavelengths. *Hence, since $\lambda_f > \lambda_g$, the wavelength of the spectral line increases toward the red end of the spectrum, and the entire process of the slowing down of clocks in a gravitational field is referred to as the* ***gravitational red shift.***

A similar analysis in terms of frequency can be obtained from equations 2.32, 2.34, and the binomial theorem equation 1.34, to yield

$$\nu_f = \nu_g\left(1 - \frac{gh}{c^2}\right) \tag{2.37}$$

Where now the frequency observed in the gravitational-free space is less than the frequency emitted in the gravitational field because the term $\left(1 - \frac{gh}{c^2}\right)$ is less than one. The change in frequency per unit frequency emitted, found from equation 2.37, is

$$\nu_f - \nu_g = -\frac{gh}{c^2}\nu_g$$

$$\frac{\nu_g - \nu_f}{\nu_g} = \frac{gh}{c^2}$$

$$\frac{\Delta\nu}{\nu_g} = \frac{gh}{c^2} \tag{2.38}$$

The gravitational red shift was confirmed on the earth by an experiment by R. V. Pound and G. A. Rebka at Harvard University in 1959 using a technique called the *Mossbauer effect.* Gamma rays were emitted from radioactive cobalt in the basement of the Jefferson Physical Laboratory at Harvard University. These gamma rays traveled 22.5 m, through holes in the floors, up to the top floor. The difference between the emitted and absorbed frequency of the gamma ray was found to agree with equation 2.38.

Example 2.5

Gravitational frequency shift. Find the change in frequency per unit frequency for a γ-ray traveling from the basement, where there is a large gravitational field, to the roof of the building, which is 22.5 m higher, where the gravitational field is weaker.

Solution

The change in frequency per unit frequency, found from equation 2.38, is

$$\frac{\Delta\nu}{\nu_g} = \frac{gh}{c^2}$$
$$= \left(9.80\ \frac{\text{m}}{\text{s}^2}\right)\left[\frac{22.5\ \text{m}}{\left(3\times10^8\ \text{m/s}\right)^2}\right]$$
$$= 2.45 \times 10^{-15}$$

Go to Interactive Example

The experiment was repeated by Pound and J. L. Snider in 1965, with another confirmation. Since then the experiment has been repeated many times, giving an accuracy to the gravitational red shift to within 1%.

Further confirmation of the gravitational red shift came from an experiment by Joseph Hafele and Richard Keating. Carrying four atomic clocks, previously synchronized with a reference clock in Washington, D.C., Hafele and Keating flew around the world in 1971. On their return they compared their airborne clocks to the clock on the ground and found the time differences associated with the time dilation effect and the gravitational effect exactly as predicted. Further tests with atomic clocks in airplanes and rockets have added to the confirmation of the gravitational red shift.

2.7 The Shapiro Experiment

Einstein's theory of general relativity not only predicts the slowing down of clocks in a gravitational field but it also predicts a contraction of the length of a rod in a gravitational field. The shrinking of rods and slowing down of clocks in a gravitational field can also be represented as a curvature of spacetime caused by mass. The slowing down of clocks and gravitational length contraction result in a reduction in the speed of light near a large massive body such as the sun. I. I. Shapiro performed an experiment in 1970 where he measured the time it takes for a radar signal (a light wave) to bounce off the planet Venus and return to earth at a time when Venus is close to the sun. The slowing down of light as it passes the sun causes the radar signal to be delayed by about 240×10^{-6} s. Shapiro's results agree with Einstein's theory to an accuracy of about 3%.

As an additional confirmation the delay in the travel time of radio signals to the spacecraft *Mariner 6* and *Mariner 7* showed the same kind of results.

Have You Ever Wondered?...
An Essay on the Application of Physics
The Black Hole

Have you ever wondered, while watching those science fiction movies, why the astronauts were afraid of a black hole? They certainly make them seem very sinister. Are they really that dangerous? What is a black hole? How is it formed? What are its characteristics? What would happen if you went into one? Is it possible to go space traveling through a black hole?

The simplest way to describe the black hole is to start with a classical analogue. Suppose we wished to launch a rocket from the earth to a far distant place in outer space. How fast must the rocket travel to escape the gravitational pull of the earth? When we launch the rocket it has a velocity v, and hence, a kinetic energy. As the rocket proceeds into space, its velocity decreases but its potential energy increases. The potential energy of an object when it is a distance r away from the center of the earth is found from

$$\text{PE} = -\frac{GM_e m}{r}$$

where G is the universal gravitational constant, M_e is the mass of the earth, and m is the mass of the object. Let us now apply this potential energy term to a rocket that is trying to escape from the gravitational pull of the earth. The total energy of the rocket at any time is equal to the sum of its potential energy and its kinetic energy, that is,

$$E = \text{KE} + \text{PE} = \tfrac{1}{2}mv^2 - \left[GM_e m\left(\frac{1}{r}\right)\right] \tag{2H.1}$$

When the rocket is fired from the surface of the earth, $r = R$, at an escape velocity v_e its total energy will be

$$E = \frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right]$$

By the law of conservation of energy, the total energy of the rocket remains a constant. Hence, we can equate the total energy at the surface of the earth to the total energy when the rocket is far removed from the earth. That is,

$$\frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right] = \frac{1}{2}mv^2 - \left[GM_e m\left(\frac{1}{r}\right)\right] \tag{2H.2}$$

When the rocket escapes the pull of the earth it has effectively traveled to infinity, that is, $r = \infty$, and its velocity at that time is reduced to zero, that is, $v = 0$. Hence, equation 2H.2 reduces to

$$\frac{1}{2}mv_e^2 - \left[GM_e m\left(\frac{1}{R}\right)\right] = 0 - \left[GM_e m\left(\frac{1}{\infty}\right)\right] = 0$$
$$\frac{1}{2}mv_e^2 = \frac{GM_e m}{R}$$
$$v_e^2 = \frac{2GM_e}{R}$$
$$v_e = \sqrt{\frac{2GM_e}{R}} \tag{2H.3}$$

Equation 2H.3 is the *escape velocity of the earth.* This is the velocity that an object must have if it is to escape the gravitational field of the earth. Now it was first observed by a British amateur astronomer, the Rev. John Michell, in 1783, and then 15 years later by Marquis Pierre de Laplace, that if light were a particle, as originally proposed by Sir Isaac Newton, then there was a limit to the size the earth could be and still have light escape from it. That is, if we solve equation 2H.3 for R, and replace the velocity of escape $v_{\mathbf{e}}$ by the velocity of light c, we get

$$R_S = \frac{2GM_{\mathbf{e}}}{c^2} \tag{2H.4}$$

For reasons that will be explained later, this value of R is called the *Schwarzschild radius,* and is designated as R_S. Solving equation 2H.4 for the Schwarzschild radius of the earth we get 8.85×10^{-3} m, which means that if the earth were contracted to a sphere of radius smaller than 8.85×10^{-3} m, then the escape velocity from the earth would be greater than the velocity of light. That is, nothing, not even light could escape from the earth if it were this small. The earth would then be called a black hole because we could not see anything coming from it.

The reason for the name, black hole, comes from the idea that if we look at an object in space, such as a star, we see light coming from that star. If the star became a black hole, no light could come from that star. Hence, when we look into space we would no longer see a bright star at that location, but rather nothing but the blackness of space. There seems to be a hole in space where the star used to be and therefore we say that there is a black hole there.

Solving equation 2H.4 for the Schwarzschild radius of the sun, by replacing the mass of the earth by the mass of the sun, we get 2.95×10^3 m. Thus, if the sun were to contract to a radius below 2.95×10^3 m the gravitational force would become so great that no light could escape from the sun, and the sun would become a black hole.

Up to this point the arguments have been strictly classical. Since Einstein's theory of general relativity is a theory of gravitation, what does it say about black holes? As we have seen, Einstein's theory of general relativity says that mass warps spacetime and we saw this in the rubber sheet analogy in figure 2.18. The greater the mass of the gravitating body the greater the warping of spacetime. Figure 1(a) shows the warping of spacetime by a star. Figure 1(b) shows the warping for a much more massive star. As the radius of the star becomes much smaller, the warping becomes more pronounced as the star approaches the size of a black hole, figure 1(c).

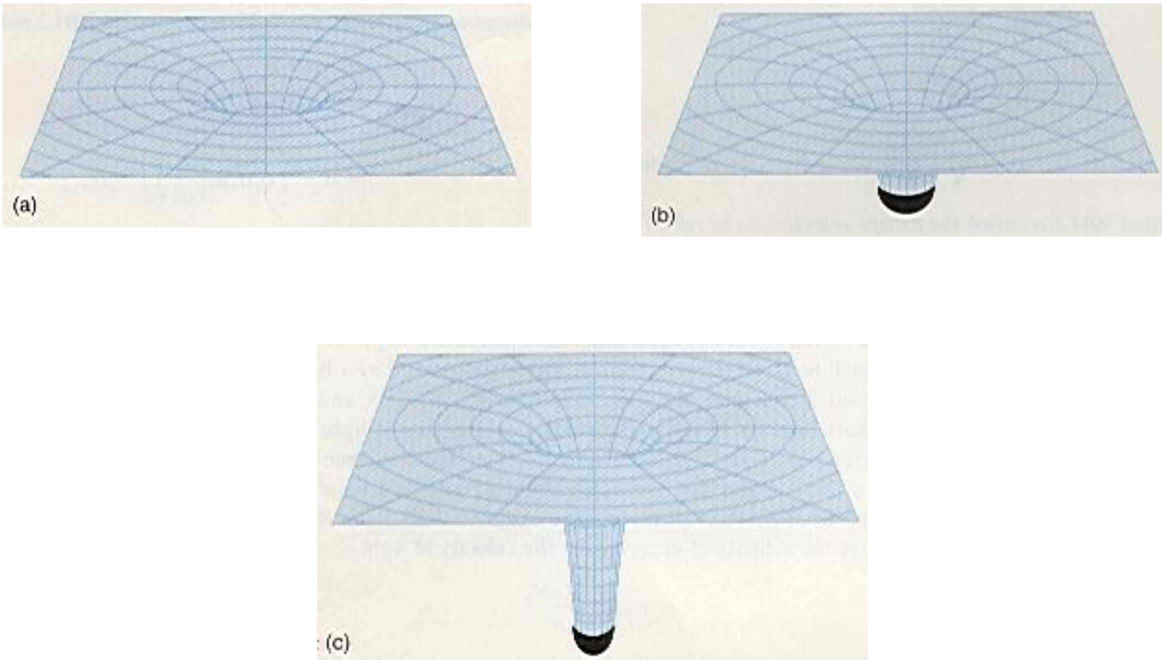

Figure 1 The warping of spacetime.

Shortly after Einstein stated his principle of general relativity, K. Schwarzschild solved Einstein's equations for the gravitational field of a point mass. For the radial portion of the solution he obtained

$$(ds)^2 = \frac{(dr)^2}{1 - 2GM/rc^2} - (1 - 2GM/rc^2)c^2(dt)^2 \tag{2H.5}$$

Equation 2H.5 is called the *radial portion of the Schwarzschild metric* and is the radial portion of the invariant interval of spacetime curved by the presence of a point mass. The invariant interval found previously in equation 2.11 is the metric for a flat spacetime, that is, one in which there is no mass to warp spacetime. That is, for flat spacetime

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

and in only one space dimension by

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

We saw there that if $ds = 0$, then $dx/dt = c$, the velocity of light, and it is a constant, hence $ds = 0$ represents the world line of a ray of light. Using the same analogy for the radial portion of the Schwarzschild solution we have

$$(ds)^2 = \frac{(dr)^2}{1 - 2GM/rc^2} - \left(1 - \frac{2GM}{rc^2}\right)c^2(dt)^2$$

As we have just seen, $ds = 0$ represents the world line of a ray of light. Applying this to the Schwarzschild solution we get

$$\frac{(dr)^2}{1 - 2GM/rc^2} = \left(1 - \frac{2GM}{rc^2}\right)c^2(dt)^2$$
$$\frac{(dr)^2}{(dt)^2} = \left(1 - \frac{2GM}{rc^2}\right)^2 c^2$$
$$\frac{dr}{dt} = \left(1 - \frac{2GM}{rc^2}\right)c \tag{2H.6}$$

Notice that if $r = 2GM/c^2$, then $dr/dt = 0$. This means that the velocity of light dr/dt is then zero, and no light is able to leave the gravitating body. But notice that this quantity is exactly what we already called the Schwarzschild radius. The Schwarzschild radius is also called the *event horizon of the black hole.* We can generalize equation 2H.6 to the form

$$\frac{dr}{dt} = \left(1 - \frac{R_s}{r}\right)c \tag{2H.7}$$

The solution of equation 2H.7 for various values of r is shown in table 2H.1. Notice that the velocity of light is not a constant near the black hole, but in a distance of only 1000 times the radius of the black hole, the velocity of light approaches the constant value c. *Note that the constancy of the velocity of light is not a postulate of general relativity as it is for special relativity*. Also note that as we get far away from the black hole, $r \gg R_s$, we enter the region of flat spacetime and the velocity of light has the constant value c of special relativity. However, within the event horizon, equation 2H.7 and table 2H.1 show that the velocity of light can be greater than c.

Table 2H.1
Variation of the Velocity of Light as a Function of the Schwarzschild Radius

r	dr/dt
$R_s/10$	$-9c$
$R_s/5$	$-4c$
$R_s/2$	$-c$
R_s	0
$2R_s$	$0.5c$
$10R_s$	$0.9c$
$100R_s$	$0.99c$
$1000R_s$	$0.999c$

The argument up to now may seem somewhat academic, in that we have described some of the characteristics of black holes, but do they really exist in nature? That is, is it possible for any objects in the universe to become black holes? The answer is yes. In the ordinary evolution of very massive stars, black holes can be formed. A star is essentially a gigantic nuclear reactor converting hydrogen to helium in a process called *nuclear fusion.* Think of the star as millions of hydrogen bombs going off at the same time, thereby producing enormous quantities of energy and enormous forces outward from the star. There is an equilibrium between the gravitational forces inward and the forces outward caused by the exploding gases. Eventually, when all the nuclear fuel is used, there is no longer an equilibrium condition. The gravitational force causes the gas to become very compact. If the star is large enough, it is compressed below its Schwarzschild radius and a black hole is formed. For an evolving star to condense into a black hole it must be approximately 25 times the mass of the sun. When the star condenses to a black hole it does not stop at the event horizon but continues to reduce in size until it becomes a singularity, a point mass. That is, the entire mass of the star has condensed to the size of a point.

There is experimental evidence that a black hole has been found as a companion of the star Cygnus X-1 and more are looked for every day.

Since time slows down in a gravitational field, the effect becomes much more pronounced in the vicinity of the black hole. If a person were to fall into the black hole he would eventually be crushed due to the enormous gravitational forces. Time would slow down for him as he approached the event horizon. At the event horizon, time would stand still for him.

The Schwarzschild black hole is an example of a nonrotating massive body. However, just as the sun and planets rotate about their axes, a more general solution of a black hole should also be concerned with the rotation of the massive body. The solution to the rotating black hole is called a *Kerr black hole,* after Roy Kerr, a New Zealand mathematician. The rotating black hole[4] (essentially an accelerating black hole) drags spacetime around with it, forming a second event horizon, thus leaving a space between the first event horizon and the second event horizon. It has been speculated that it may be possible to enter the first event horizon, but not the second, and exit somewhere else in either another universe or in this universe in another place and/or time.

It has also been speculated that there might also exist white holes in space. That is, mass is drawn into a black hole, but would be spewed out of a white hole. In fact some physicists have speculated that a black hole in one universe is a white hole in another universe.

4[4]See interactive tutorial problem 15.

The Language of Physics

Spacetime diagram
A graph of a particle's space and time coordinates. The time coordinate is usually expressed as τ, which is equal to the product of the speed of light and the time (p.).

World line
A line in a spacetime diagram that shows the motion of a particle through spacetime. A world line of a particle at rest or moving at a constant velocity is a straight line in spacetime. The world line of a light ray makes an angle of 45^0 with the τ-axis in spacetime. The world line of an accelerated particle is a curve in spacetime (p.).

Light cone
A cone that is drawn in spacetime showing the relation between the past and the future of a particle in spacetime. World lines within the cone are called timelike because they are accessible to us in time. Events outside the cone are called spacelike because they occur in another part of space that is not accessible to us and hence is called elsewhere (p.).

Invariant interval
A constant value in spacetime that all observers agree on, regardless of their state of motion. The equation of the invariant interval is in the form of a hyperbola in spacetime. Because of the hyperbolic form of the invariant interval, Euclidean geometry does not hold in spacetime. The reason for length contraction and time dilation is the fact that spacetime is non-Euclidean. The longest distance in spacetime is the straight line (p.).

Equivalence principle
On a local scale, the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system. Hence, an accelerated frame of reference is equivalent to an inertial frame of reference in which gravity is present, and an inertial frame is equivalent to an accelerated frame in which gravity is absent (p.).

The general theory of relativity
The laws of physics are the same in all frames of reference (note that there is no statement about the constancy of the velocity of light as in the special theory of relativity) (p.).

Warped spacetime
Matter causes spacetime to be warped so that the world lines of particles in spacetime are curved. Hence, matter warps spacetime and spacetime tells matter how to move. Gravity is a consequence of the warping of spacetime by matter (p.).

Gravitational red shift
Time elapsed on a clock in a gravitational field is less than the time elapsed on a clock in a gravity-free space. This effect of the slowing down of a clock in a gravitational field is manifested by observing a spectral line from an excited atom in a gravitational field. The wavelength of the spectral line of that atom is shifted toward the red end of the electromagnetic spectrum (p.).

Summary of Important Equations

Tau in spacetime $\tau = ct$ (2.1)

Velocity in a spacetime diagram $v = c \tan\theta$ (2.2)

The square of the invariant interval

$$(ds)^2 = c^2(dt)^2 - (dx)^2 \tag{2.10}$$

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{2.11}$$

$$(ds)^2 = (d\tau)^2 - (dx)^2 \tag{2.13}$$

Slowing down of a clock in a gravitational field

$$\Delta t_f = \Delta t_g\left(1 + \frac{gh}{c^2}\right) \tag{2.31}$$

Gravitational red shift of wavelength

$$\lambda_f = \lambda_g\left(1 + \frac{gh}{c^2}\right) \tag{2.35}$$

Gravitational red shift of frequency

$$\nu_f = \nu_g\left(1 - \frac{gh}{c^2}\right) \tag{2.37}$$

Change in frequency per unit frequency

$$\frac{\Delta\nu}{\nu_g} = \frac{gh}{c^2} \tag{2.38}$$

Questions for Chapter 2

1. Discuss the concept of spacetime. How is it like space and how is it different?
2. How many light cones are there in your classroom?
3. Why can't a person communicate with another person who is elsewhere?
*4. Discuss the twin paradox on the basis of figure 2.6(b).
5. Using figure 2.7, discuss why the scales in the S' system are not the same as the scales in the S system.
*6. Considering some of the characteristics of spacetime, that is, it can be warped, and so forth, could spacetime be the elusive ether?

7. What does it mean to say that spacetime is warped?
8. Describe length contraction by a spacetime diagram.
9. Describe time dilation by a spacetime diagram.
10. Discuss simultaneity with the aid of a spacetime diagram.

Problems for Chapter 2

2.1 Spacetime Diagrams

1. Draw the world line in spacetime for a particle moving in (a) an elliptical orbit, (b) a parabolic orbit, and (c) a hyperbolic orbit.

2.2 The Invariant Interval

2. Find the angle that the world line of a particle moving at a speed of $c/4$ makes with the τ-axis in spacetime.

3. The world line of a particle is a straight line making an angle of 30^0 below the τ-axis. Determine the speed of the particle.

4. The world line of a particle is a straight line of length 150 m. Find the value of dx if $d\tau = 200$ m.

5. (a) On a sheet of graph paper draw the hyperbolas representing the invariant interval of spacetime as shown in figure 2.7. (b) Draw the S'-axes on this diagram for a particle moving at a speed of $c/4$.

6. Using the graph of problem 5, draw a rod 1.50 units long at rest in the S frame of reference. (a) From the graph determine the length of the rod in the S' frame of reference. (b) Determine the length of the rod using the Lorentz contraction equation.

7. Using the graph of problem 5, draw a rod 1.50 units long at rest in the S' frame of reference. (a) From the graph determine the length of the rod in the S frame of reference. (b) Determine the length of the rod using the Lorentz contraction equation.

2.6 The Gravitational Red Shift

8. One twin lives on the ground floor of a very tall apartment building, whereas the second twin lives 61.0 m above the ground floor. What is the difference in their age after 50 years?

9. The lifetime of a subatomic particle is 6.25×10^{-7} s on the earth's surface. Find its lifetime at a height of 500 km above the earth's surface.

10. An atom on the surface of Jupiter ($g = 23.1$ m/s^2) emits a ray of light of wavelength 528.0 nm. What wavelength would be observed at a height of 10,000 m above the surface of Jupiter?

Additional Problems

*11. Using the principle of equivalence, show that the difference in time between a clock at rest and an accelerated clock should be given by

$$\Delta t_R = \Delta t_A\left(1 + \frac{ax}{c^2}\right)$$

where $\Delta t_{\mathbf{R}}$ is the time elapsed on a clock at rest, $\Delta t_{\mathbf{A}}$ is the time elapsed on the accelerated clock, a is the acceleration of the clock, and x is the distance that the clock moves during the acceleration.

*12. A particle is moving in a circle of 1.00-m radius and undergoes a centripetal acceleration of 9.80 m/s^2. Using the results of problem 11, determine how many revolutions the particle must go through in order to show a 10% variation in time.

13. The pendulum of a grandfather clock has a period of 0.500 s on the surface of the earth. Find its period at an altitude of 200 km. *Hint:* Note that the change in the period is due to two effects. The acceleration due to gravity is smaller at this height even in classical physics, since

$$g = \frac{GM}{(R + h)^2}$$

To solve this problem, use the fact that the average acceleration is

$$g = \frac{GM}{R(R + h)}$$

and assume that

$$\Delta t_f = \Delta t_g\left(1 + \frac{gh}{c^2}\right)$$

14. Compute the fractional change in frequency of a spectral line that occurs between atomic emission on the earth's surface and that at a height of 325 km.

Interactive Tutorials

15. *A rotating black hole.* Assume the sun were to collapse to a black hole as described in the "Have you ever wondered … ?" section. (a) Calculate the radius of the black hole, which is called the Schwarzschild radius R_{S}. Since the sun is also rotating, angular momentum must be conserved. Therefore as the sun collapses the angular velocity of the sun must increase, and hence the tangential velocity of a point on the surface of the sun must also increase. (b) Find the radius of the sun during the collapse such that the tangential velocity of a point on the equator is equal to the velocity of light c. Compare this radius to the Schwarzschild radius. Some characteristics of the sun are radius, $r_0 = 6.96 \times 10^8$ m, mass of sun $M = 1.99 \times 10^{30}$ kg, and the angular velocity of the sun $\omega_0 = 2.86 \times 10^{-6}$ rad/s.

16. *Gravitational red shift.* An atom on the surface of the earth emits a ray of light of wavelength $\lambda_{\mathbf{g}} = 528.0$ nm, straight upward. (a) What wavelength $\lambda_{\mathbf{f}}$ would be observed at a height $y = 10{,}000$ m? (b) What frequency $\nu_{\mathbf{f}}$ would be observed at this height? (c) What change in time would this correspond to?

Go to Interactive Tutorials

Chapter 3 Quantum Physics

"Newton himself was better aware of the weakness inherent in his intellectual edifice than the generations which followed him. This fact has always aroused my admiration." Albert Einstein

3.1 The Particle Nature of Waves

Up to now in our study of physics, we considered (1) the motion of particles and their interaction with other particles and their environment and (2) the nature, representation, and motion of waves. We considered particles as little hard balls of matter while a wave was a disturbance that was spread out through a medium. There was certainly a significant difference between the two concepts, and one of the most striking of these is illustrated in figure 3.1. In figure 3.1(a), two

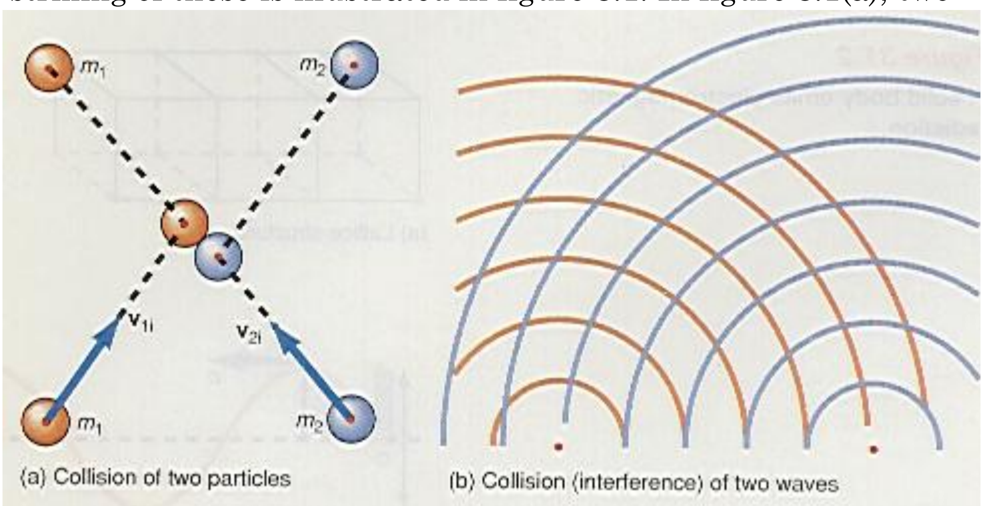

Figure 3.1 Characteristics of particles and waves.

particles collide, bounce off each other, and then continue in a new direction. In figure 3.1(b), two waves collide, but they do not bounce off each other. They add together by the principle of superposition, and then each continues in its original direction as if the waves never interacted with each other.

Another difference between a particle and a wave is that the total energy of the particle is concentrated in the localized mass of the particle. In a wave, on the other hand, the energy is spread out throughout the entire wave. Thus, there is a very significant difference between a particle and a wave.

We have seen that light is an electromagnetic wave. The processes of interference, diffraction, and polarization are characteristic of wave phenomena and have been studied and verified in the laboratory many times over. Yet there has appeared with time, some apparent contradictions to the wave nature of light. We will discuss the following three of these physical phenomena:

1. Blackbody radiation.
2. The photoelectric effect.

3. Compton scattering.

3.2 Blackbody Radiation

All bodies emit and absorb radiation. (Recall that radiation is heat transfer by electromagnetic waves.) The Stefan-Boltzmann law showed that the amount of energy radiated is proportional to the fourth power of the temperature, but did not say how the heat radiated was a function of the wavelength of the radiation. Because the radiation consists of electromagnetic waves, we would expect that the energy should be distributed evenly among all possible wavelengths. However, the energy distribution is not even but varies according to wavelength and frequency. All attempts to account for the energy distribution by classical means failed.

Let us consider for a moment how a body can radiate energy. We know that an oscillating electric charge generates an electromagnetic wave. A body can be considered to be composed of a large number of atoms in a lattice structure as shown in figure 3.2(a). For a metallic material the positively ionized atom is located

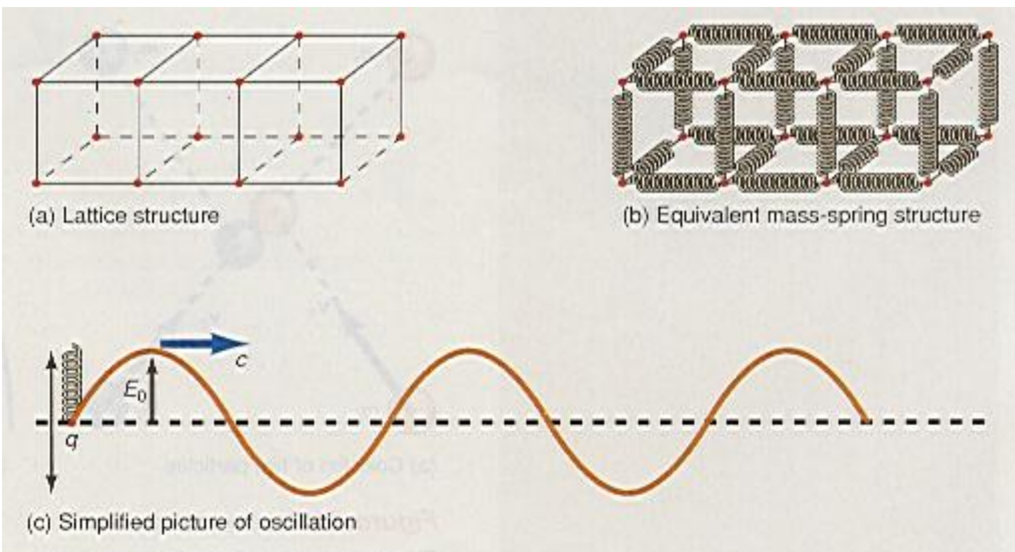

Figure 3.2 A solid body emits electromagnetic radiation.

at the lattice site and the outermost electron of the atom moves throughout the lattice as part of the electron gas. Each atom of the lattice is in a state of equilibrium under the action of all the forces from all its neighboring atoms. The atom is free to vibrate about this equilibrium position. A mechanical analogue to the lattice structure is shown in figure 3.2(b) as a series of masses connected by springs. Each mass can oscillate about its equilibrium position. To simplify the picture further, let us consider a single ionized atom with a charge q and let it oscillate in simple harmonic motion, as shown in figure 3.2(c). The oscillating charge generates an electromagnetic wave that is emitted by the body. Each ionized atom is an oscillator and each has its own fixed frequency and emits radiation of this frequency. Because the body is made up of millions of these oscillating charges, the body always emits radiation of all these different frequencies, and hence the emission spectrum should be continuous. The intensity of the radiation depends on the amplitude of the oscillation. As you recall from general physics, a typical radiated wave is given by

$$E = E_0 \sin(kx - \omega t)$$

where

$$k = \frac{2\pi}{\lambda} \tag{12.9}$$

and

$$\omega = 2\pi\nu \tag{12.12}$$

Thus, the frequency of the oscillating charge is the frequency of the electromagnetic wave. The amplitude of the wave E_0 depends on the amplitude of the simple harmonic motion of the oscillating charge. When the body is heated, the heat energy causes the ionized atoms to vibrate with greater amplitude about their equilibrium position. The energy density of the emitted waves is given by

$$u = \varepsilon_0 E^2$$

or

$$u = \varepsilon_0 E_0^2 \sin^2(kx - \omega t) \tag{3.1}$$

Thus, when the amplitude of the oscillation E_0 increases, more energy is emitted. When the hot body is left to itself it loses energy to the environment by this radiation process and the amplitude of the oscillation decreases. The amplitude of the oscillation determines the energy of the electromagnetic wave. Because of the extremely large number of ionized atoms in the lattice structure that can participate in the oscillations, all modes of vibration of the lattice structure are possible and hence all possible frequencies are present. Thus, the classical picture of blackbody radiation permits all frequencies and energies for the electromagnetic waves. However, this classical picture does not agree with experiment.

Max Planck (1858-1947), a German physicist, tried to "fit" the experimental results to the theory. However, he found that he had to break with tradition and propose a new and revolutionary concept. *Planck assumed that the atomic oscillators cannot take on all possible energies, but could only oscillate with certain discrete amounts of energy given by*

$$E = nh\nu \tag{3.2}$$

where h is a constant, now called *Planck's constant,* and has the value

$$h = 6.625 \times 10^{-34} \text{ J s}$$

In equation 3.2 ν is the frequency of the oscillator and n is an integer, a number, now called a *quantum number.* The energies of the vibrating atom are now said to be quantized, or limited to only those values given by equation 3.2. Hence, the atom can have energies $h\nu$, $2h\nu$, $3h\nu$, and so on, but never an energy such as 2.5 $h\nu$. This concept of quantization is at complete variance with classical electromagnetic theory. In the classical theory, as the oscillating charge radiates energy it loses energy and the amplitude of the oscillation decreases continuously. If the energy of the oscillator is quantized, the amplitude cannot decrease continuously and hence the oscillating charge cannot radiate while it is in this quantum state. If the oscillator now drops down in energy one quantum state, the difference in energy between the two states is now available to be radiated away. Hence, *the assumption of discrete energy states entails that the radiation process can only occur when the oscillator jumps from one quantized energy state to another quantized energy state.* As an example, if the oscillating charge is in the quantum state 4 it has an energy

$$E_4 = 4h\nu$$

When the oscillator drops to the quantum state 3 it has the energy

$$E_3 = 3h\nu$$

When the oscillator drops from the 4 state to the 3 state it can emit the energy

$$\Delta E = E_4 - E_3 = 4h\nu - 3h\nu = h\nu$$

Thus, the amount of energy radiated is always in small bundles of energy of amount $h\nu$. *This little bundle of radiated electromagnetic energy was called a quantum of energy. Much later, this bundle of electromagnetic energy came to be called a* ***photon.***

Although this quantum hypothesis led to the correct formulation of blackbody radiation, it had some serious unanswered questions. Why should the energy of the oscillator be quantized? If the energy from the blackbody is emitted as a little bundle of energy how does it get to be spread out into Maxwell's electromagnetic wave? How does the energy, which is spread out in the wave, get compressed back into the little quantum of energy so it can be absorbed by an atomic oscillator? These and other questions were very unsettling to Planck and the physics community in general. Although Planck started what would be eventually called *quantum mechanics,* and won the Nobel Prize for his work, he spent many years trying to disprove his own theory.

Example 3.1

Applying the quantum condition to a vibrating spring. A weightless spring has a spring constant k of 29.4 N/m. A mass of 300 g is attached to the spring and is then displaced 5.00 cm. When the mass is released, find (a) the total energy of the mass, (b) the frequency of the vibration, (c) the quantum number n associated with this energy, and (d) the energy change when the oscillator changes its quantum state by one value, that is, for $n = 1$.

Solution

a. The total energy of the vibrating spring comes from its potential energy, which it obtained when work was done to stretch the spring to give an amplitude A of 5.00 cm. The energy, with $x = A$, is

$$E_{\text{total}} = \text{PE} = \frac{1}{2}kA^2$$

$$E = \frac{1}{2}(29.4\ \text{N/m})(0.0500\ \text{m})^2$$

$$= 3.68 \times 10^{-2}\ \text{J}$$

b. The frequency ν of the vibration, is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{29.4\ \text{N/m}}{0.300\ \text{kg}}}$$

$$= 1.58\ \text{Hz}$$

c. The quantum number n associated with this energy, found from equation 3.2, is

$$E = nh\nu$$

$$n = \frac{E}{h\nu}$$

$$= \frac{3.68 \times 10^{-2}\ \text{J}}{6.625 \times 10^{-34}\ \text{J s} \times 1.58\ \text{s}^{-1}}$$

$$= 3.52 \times 10^{31}$$

This is an enormously large number. Therefore, the effect of a quantum of energy is very small unless the vibrating system itself is very small, as in the case of the vibration of an atom.

d. The energy change associated with the oscillator changing one energy state, found from equation 3.2, is

$$E = nh\nu = h\nu$$

$$= (6.625 \times 10^{-34}\ \text{J s})(1.58\ \text{s}^{-1})$$

$$= 1.05 \times 10^{-33}\ \text{J}$$

This change in energy is so small that for all intents and purposes, the energy of a vibrating spring-mass system is continuous.

Go to Interactive Example

Example 3.2

The energy of a photon of light. An atomic oscillator emits radiation of 700.0-nm wavelength. How much energy is associated with a photon of light of this wavelength?

Solution

The energy of the photon, given by equation 3.2, is

$$E = h\nu$$

but since the frequency ν can be written as c/λ, the energy of the photon can also be written as

$$E = h\nu = \frac{hc}{\lambda}$$

$$= \frac{(6.625\times10^{-34}\ \text{J s})(3.00\times10^{8}\ \text{m/s})}{700.0\ \text{nm}}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$

$$= 2.84 \times 10^{-19}\ \text{J}$$

Thus, the photon of light is indeed a small bundle of energy.

Go to Interactive Example

3.3 The Photoelectric Effect

When Heinrich Hertz performed his experiments in 1887 to prove the existence of electromagnetic waves, he accidentally found that when light fell on a metallic surface, the surface emitted electrical charges. *This effect, whereby light falling on a metallic surface produces electrical charges, is called the* ***photoelectric effect.*** The photoelectric effect was the first proof that light consists of small particles called photons. Thus, the initial work that showed light to be a wave would also show that light must also be a particle.

Further experiments by Philipp Lenard in 1900 confirmed that these electrical charges were electrons. These electrons were called *photoelectrons*. The photoelectric effect can best be described by an experiment, the schematic diagram of which is shown in figure 3.3. The switch S is thrown to make the anode of the phototube positive and the cathode negative. Monochromatic light (light of a single frequency ν) of intensity I_1, is allowed to shine on the cathode of the phototube, causing electrons to be emitted.

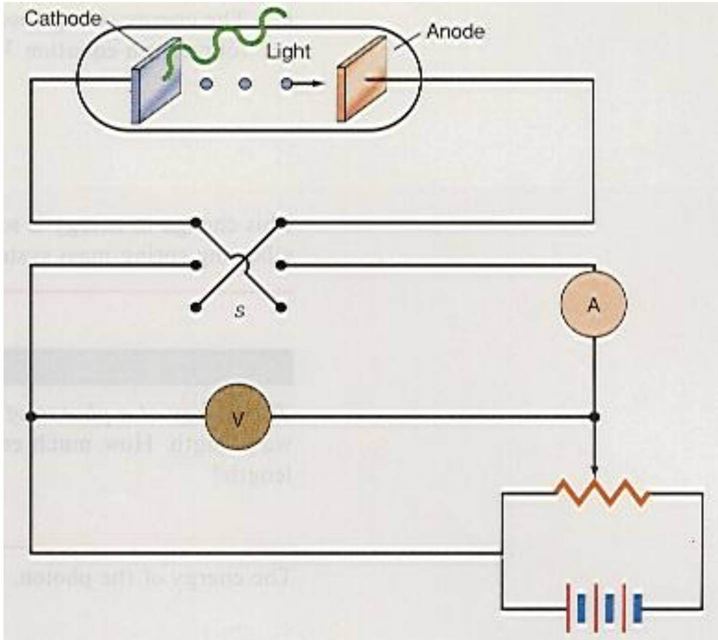

Figure 3.3 Schematic diagram for the photoelectric effect.

The positive anode attracts these electrons, and they flow to the anode and then through the connecting circuit. The ammeter in the circuit measures this current. Starting with a positive potential V, the current is observed for decreasing values of V. When the potential V is reduced to zero, the switch S is reversed to make the anode negative and the cathode positive. The negative anode now repels the photoelectrons as they approach the anode. If this potential is made more and more negative, however, a point is eventually reached when the kinetic energy of the electrons is not great enough to overcome the negative stopping potential, and no more electrons reach the anode. The current i, therefore, becomes zero. A plot of the current i in the circuit, as a function of the potential between the plates, is shown in figure 3.4. If we increase the intensity of the light to I_2 and repeat the experiment, we obtain the second curve shown in the figure.

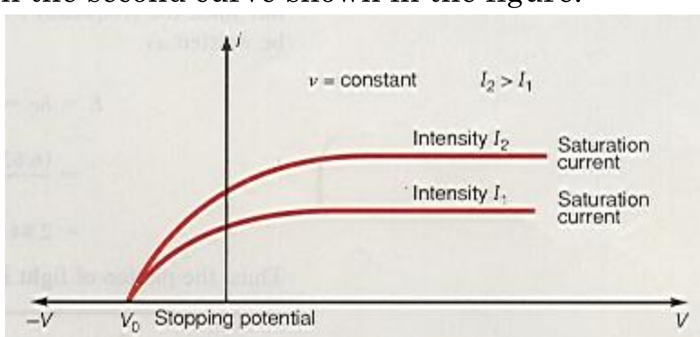

Figure 3.4 Current i as a function of voltage V for the photoelectric effect.

An analysis of figure 3.4 shows that when the value of V is high and positive, the current i is a constant. This occurs because all the photoelectrons formed at the cathode are reaching the anode. By increasing the intensity I, we obtain a higher constant value of current, because more photoelectrons are being emitted per unit time. This shows that the number of electrons emitted (the current) is proportional to the intensity of the incident light, that is,

$$i \propto I$$

Notice that when the potential is reduced to zero, there is still a current in the tube. Even though there is no electric field to draw them to the anode, many of the photoelectrons still reach the anode because of the initial kinetic energy they possess when they leave the cathode. As the switch S in figure 3.3 is reversed, the potential V between the plates becomes negative and tends to repeal the photoelectrons. As the retarding potential V is made more negative, the current i (in figure 3.4) decreases, indicating that fewer and fewer photoelectrons are reaching the anode. When V is reduced to V_0, there is no current at all in the circuit; V_0 is called the *stopping potential.* Note that it is the same value regardless of the intensity. (Both curves intersect at V_0.) Hence the stopping potential is independent of the intensity of light, or stated another way, the stopping potential is not a function of the intensity of light. Stated mathematically this becomes,

$$V_0 \neq V_0(I) \tag{3.3}$$

The retarding potential is related to the kinetic energy of the photoelectrons. For the electron to reach the anode, its kinetic energy must be equal to the potential energy between the plates. (A mechanical analogy might be helpful at this point. If we wish to throw a ball up to a height h, where it will have the potential energy PE = mgh, we must throw the ball with an initial velocity v_0 such that the initial kinetic energy of the ball $\text{KE} = \frac{1}{2}mv_0^2$, is equal to the final potential energy of the ball.) Hence, the kinetic energy of the electron must be

$$\text{KE of electron} = \text{PE between the plates}$$

or

$$\text{KE} = eV \tag{3.4}$$

where e is the charge on the electron and V is the potential between the plates.

The retarding potential acts on electrons that have less kinetic energy than that given by equation 3.4. When $V = V_0$, the stopping potential, even the most energetic electrons (those with maximum kinetic energy) do not reach the anode. Therefore,

$$\text{KE}_{\textbf{max}} = eV_0 \tag{3.5}$$

As equations 3.3 and 3.5 show, the maximum kinetic energy of the photoelectrons is not a function of the intensity of the incident light, that is,

$$\text{KE}_{\textbf{max}} \neq \text{KE}_{\textbf{max}}(I)$$

It is also found experimentally that there is essentially no time lag between the time the light shines on the cathode and the time the photoelectrons are emitted.

If we keep the intensity constant and perform the experiment with different frequencies of light, we obtain the curves shown in figure 3.5. As the graph in figure 3.5 shows, the saturation current (the maximum current) is the same for

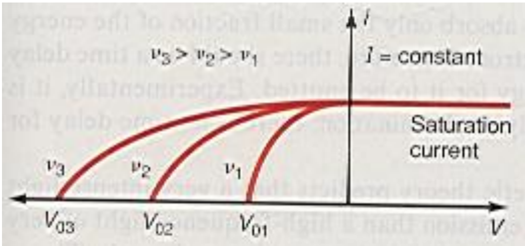

Figure 3.5 Current i as a function of voltage V for different light frequencies.

any frequency of light, as long as the intensity is constant. But the stopping potential is different for each frequency of the incident light. Since the stopping potential is proportional to the maximum kinetic energy of the photoelectrons by equation 3.5, the maximum kinetic energy of the photoelectrons should be proportional to the frequency of the incident light. The maximum kinetic energy of the photoelectrons is plotted as a function of frequency in the graph of figure 3.6.

The first thing to observe is that the maximum kinetic energy of the photoelectrons is proportional to the frequency of the incident light. That is,

$$KE_{\mathbf{max}} \propto \nu$$

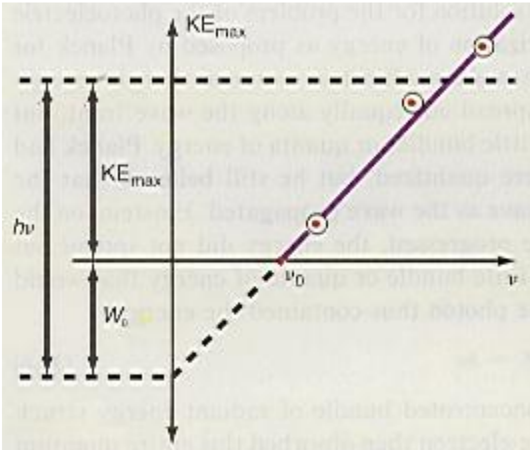

Figure 3.6 Maximum kinetic energy (KE_{max}) as a function of frequency ν for the photoelectric effect.

The second thing to observe is that there is a cutoff frequency ν_0 below which there is no photoelectronic emission. That is, no photoelectric effect occurs unless the incident light has a frequency higher than the threshold frequency ν_0. For most metals, ν_0 lies in the ultraviolet region of the spectrum, but for the alkali metals it lies in the visible region.

Failure of the Classical Theory of Electromagnetism to Explain the Photoelectric Effect

The classical theory of electromagnetism was initially used to try to explain the results of the photoelectric effect. The results of the experiment are compared with the predictions of classical electromagnetic theory in table 3.1. The only agreement

Table 3.1
The Photoelectric Effect

Experimental Results	Theoretical Predictions of Classical Electromagnetism	Agreement
$i \propto I$	$i \propto I$	Yes
Cutoff frequency ν_0	There should not be a cutoff frequency	No
No time lag for emission of electrons	There should be a time lag	No
$KE_{max} \propto \nu$	KE_{max} not $\propto \nu$	No
$KE_{max} \neq KE_{max}(I)$	$KE_{max} \propto I$	No

between theory and experiment is the fact that the photocurrent is proportional to the intensity of the incident light. According to classical theory, there should be no minimum threshold frequency ν_0 for emission of photoelectrons. This prediction does not agree with the experimental results.

According to classical electromagnetic theory, energy is distributed equally throughout the entire electric wave front. When the wave hits the electron on the cathode, the electron should be able to absorb only the small fraction of the energy of the total wave that is hitting the electron. Therefore, there should be a time delay to let the electron absorb enough energy for it to be emitted. Experimentally, it is found that emission occurs immediately on illumination; there is no time delay for emission.

Finally, classical electromagnetic theory predicts that a very intense light of very low frequency will cause more emission than a high-frequency light of very low intensity. Again the theory fails to agree with the experimental result. Therefore, classical electromagnetic theory cannot explain the photoelectric effect.

Einstein's Theory of the Photoelectric Effect

In the same year that Einstein published his special theory of relativity, 1905, he also proposed a new and revolutionary solution for the problem of the photoelectric effect. Using the concept of the quantization of energy as proposed by Planck for the solution to the blackbody radiation problem, Einstein assumed that the energy of the electromagnetic wave was not spread out equally along the wave front, but that it was concentrated into Planck's little bundles or quanta of energy. Planck had assumed that the atomic radiators were quantized, but he still believed that the energy became spread out across the wave as the wave propagated. Einstein, on the other hand, assumed that as the wave progressed, the energy did not spread out with the wave front, but stayed in the little bundle or quanta of energy that would later become known as the photon. The photon thus contained the energy

$$E = h\nu \tag{3.6}$$

Einstein assumed that this concentrated bundle of radiant energy struck an electron on the metallic surface. The electron then absorbed this entire quantum of energy ($E = h\nu$). A portion of this energy is used by the electron to break away from the solid, and the rest shows up as the kinetic energy of the electron. That is,

$$\begin{aligned}&\text{(incident absorbed energy)} - \text{(energy to break away from solid)}\\ &\qquad = \text{(maximum KE of electron)}\end{aligned} \tag{3.7}$$

We call the energy for the electron to break away from the solid the *work function of the solid* and denote it by W_0. We can state equation 3.7 mathematically as

$$E - W_0 = \text{KE}_{\mathbf{max}} \tag{3.8}$$

or

$$h\nu - W_0 = \text{KE}_{\mathbf{max}} \tag{3.9}$$

We find the final maximum kinetic energy of the photoelectrons from equation 3.9 as

$$\text{KE}_{\mathbf{max}} = h\nu - W_0 \tag{3.10}$$

Equation 3.10 is known as Einstein's photoelectric equation.

Notice from figure 3.6, when the $\text{KE}_{\mathbf{max}}$ of the photoelectrons is equal to zero, the frequency ν is equal to the cutoff frequency ν_0. Hence, equation 3.10 becomes

$$0 = h\nu_0 - W_0$$

Thus, we can also write the work function of the metal as

$$W_0 = h\nu_0 \tag{3.11}$$

Hence, we can also write Einstein's photoelectric equation as

$$\text{KE}_{\mathbf{max}} = h\nu - h\nu_0 \tag{3.12}$$

For light frequencies equal to or less than ν_0, there is not enough energy in the incident wave to remove the electron from the solid, and hence there is no photoelectric effect. This explains why there is a threshold frequency below which there is no photoelectric effect.

When Einstein proposed his theory of the photoelectric effect, there were not enough quantitative data available to prove the theory. In 1914, R. A. Millikan performed experiments (essentially the experiment described here) that confirmed Einstein's theory of the photoelectric effect.

Einstein's theory accounts for the absence of a time lag for photoelectronic emission. As soon as the electron on the metal surface is hit by a photon, the electron absorbs enough energy to be emitted immediately. Einstein's equation also correctly predicts the fact that the maximum kinetic energy of the photoelectron is dependent on the frequency of the incident light. Thus, Einstein's equation completely predicts the experimental results.

Einstein's theory of the photoelectric effect is outstanding because it was the first application of quantum concepts. *Light should be considered as having not only a wave character, but also a particle character. (The photon is the light particle.)*

For his explanation of the photoelectric effect, Einstein won the Nobel Prize in physics in 1921. As mentioned earlier, Einstein's paper on the photoelectric effect was also published in 1905 around the same time as his paper on special relativity. Thus, he was obviously thinking about both concepts at the same time. It is no wonder then that he was not too upset with dismissing the concept of the ether for the propagation of electromagnetic waves. Because he could now picture light as a particle, a photon, he no longer needed a medium for these waves to propagate in.

Example 3.3

The photoelectric effect. Yellow light of 577.0-nm wavelength is incident on a cesium surface. It is found that no photoelectrons flow in the circuit when the cathode-anode voltage drops below 0.250 V. Find (a) the frequency of the incident photon, (b) the initial energy of the photon, (c) the maximum kinetic energy of the photoelectron, (d) the work function of cesium, (e) the threshold frequency, and (f) the corresponding threshold wavelength.

Solution

a. The frequency of the photon is found from

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00 \times 10^{8}\ \text{m/s}}{577.0\ \text{nm}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 5.20 \times 10^{14}\ \text{Hz}$$

b. The energy of the incident photon, found from equation 3.6, is

$$E = h\nu = (6.625 \times 10^{-34}\ \text{J s})(5.20 \times 10^{14}\ \text{s}^{-1})$$
$$= 3.45 \times 10^{-19}\ \text{J}$$

c. The maximum kinetic energy of the photoelectron, found from equation 3.5, is

$$\begin{aligned} KE_{\mathbf{max}} &= eV_0 \\ &= (1.60 \times 10^{-19}\ \text{C})(0.250\ \text{V}) \\ &= 4.00 \times 10^{-20}\ \text{J} \end{aligned}$$

d. The work function of cesium is found by rearranging Einstein's photoelectric equation, 3.8, as

$$\begin{aligned} W_0 &= E - KE_{\mathbf{max}} \\ &= 3.45 \times 10^{-19}\ \text{J} - 4.00 \times 10^{-20}\ \text{J} \\ &= 3.05 \times 10^{-19}\ \text{J} \\ &= 1.91\ \text{eV} \end{aligned}$$

e. The threshold frequency is found by solving equation 3.11 for ν_0. Thus,

$$\begin{aligned} \nu_0 &= \frac{W_0}{h} = \frac{3.05 \times 10^{-19}\ \text{J}}{6.625 \times 10^{-34}\ \text{J s}} \\ &= 4.60 \times 10^{14}\ \text{Hz} \end{aligned}$$

f. The wavelength of light associated with the threshold frequency is found from

$$\begin{aligned} \lambda_0 &= \frac{c}{\nu_0} = \left(\frac{3.00 \times 10^8\ \text{m/s}}{4.60 \times 10^{14}\ \text{s}^{-1}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right) \\ &= 653\ \text{nm} \end{aligned}$$

This wavelength lies in the red portion of the visible spectrum.

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3.4 The Properties of the Photon

According to classical physics light must be a wave. But the results of the photoelectric effect require light to be a particle, a photon. What then is light? Is it a wave or is it a particle?

If light is a particle then it must have some of the characteristics of particles, that is, it should possess mass, energy, and momentum. Let us first consider the mass of the photon. The relativistic mass of a particle was given by equation 1.86 as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

But the photon is a particle of light and must therefore move at the speed of light c. Hence, its mass becomes

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{0} \tag{3.13}$$

But division by zero is undefined. The only way out of this problem is to *define the rest mass of a photon as being zero,* that is,

$$\textit{Photon} \qquad m_0 = 0 \tag{3.14}$$

At first this may seem a contradiction, but since the photon always moves at the speed c, it is never at rest, and therefore does not need a rest mass. With $m_0 = 0$, equation 3.13 becomes 0/0, which is an indeterminate form. Although the mass of the photon still cannot be defined by equation 1.86 it can be defined from equation 1.100, namely

$$E = mc^2$$

Hence,

$$m = \frac{E}{c^2} \tag{3.15}$$

The energy of the photon was given by

$$\textit{Energy of Photon} \qquad E = h\nu \tag{3.6}$$

Therefore, the mass of the photon can be found by substituting equation 3.6 into equation 3.15, that is,

$$\textit{Mass of Photon} \qquad m = \frac{E}{c^2} = \frac{h\nu}{c^2} \tag{3.16}$$

Example 3.4

The mass of a photon. Find the mass of a photon of light that has a wavelength of (a) 380.0 nm and (b) 720.0 nm.

Solution

a. For $\lambda = 380.0$ nm, the frequency of the photon is found from

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00 \times 10^8 \text{ m/s}}{380.0 \text{ nm}}\right)\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)$$
$$= 7.89 \times 10^{14} \text{ Hz}$$

Now we can find the mass from equation 3.16 as

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{(6.625\times10^{-34}\ \text{J s})(7.89\times10^{14}\ \text{s}^{-1})}{(3.00\times10^{8}\ \text{m/s})^2}\left[\frac{(\text{kg m/s}^2)\ \text{m}}{\text{J}}\right]$$
$$= 5.81 \times 10^{-36}\ \text{kg}$$

b. For λ = 720.0 nm, the frequency is

$$\nu = \frac{c}{\lambda} = \left(\frac{3.00\times10^{8}\ \text{m/s}}{720.0\ \text{nm}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 4.17 \times 10^{14}\ 1/\text{s}$$

and the mass is

$$m = \frac{h\nu}{c^2} = \frac{(6.625\times10^{-34}\ \text{J s})(4.17\times10^{14}\ \text{s}^{-1})}{(3.00\times10^{8}\ \text{m/s})^2}$$
$$= 3.07 \times 10^{-36}\ \text{kg}$$

As we can see from these examples, the mass of the photon for visible light is very small.

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The momentum of the photon can be found as follows. Starting with the relativistic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{1.86}$$

we square both sides of the equation and obtain

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$
$$m^2 - \frac{m^2v^2}{c^2} = m_o^2 \tag{3.17}$$

Multiplying both sides of equation 3.17 by c^4, we obtain

$$m^2c^4 - m^2v^2c^2 = m_o^2c^4$$

But $m^2c^4 = E^2$, $m_o^2c^4 = E_0^2$, and $m^2v^2 = p^2$, thus,

$$E^2 - p^2c^2 = E_0^2 \tag{3.18}$$

Hence, we find the momentum of any particle from equation 3.18 as

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} \tag{3.19}$$

For the special case of a particle of zero rest mass, $E_0 = m_0c^2 = 0$, and *the momentum of a photon,* found from equation 3.19, is

Momentum of Photon
$$p = \frac{E}{c} \tag{3.20}$$

Using equation 3.6, we can write the momentum of a photon in terms of its frequency as

$$p = \frac{E}{c} = \frac{h\nu}{c}$$

Since $\nu/c = 1/\lambda$, this is also written as

Momentum of Photon
$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.21}$$

Example 3.5

The momentum of a photon. Find the momentum of visible light for (a) λ = 380.0 nm and (b) λ = 720.0 nm.

Solution

a. The momentum of the photon, found from equation 3.21, is

$$p = \frac{h}{\lambda} = \left(\frac{(6.625 \times 10^{-34} \text{ J s})}{(380.0 \text{ nm})}\right)\left(\frac{1 \text{ nm}}{10^{-9} \text{ m}}\right)\left[\frac{(\text{kg m/s}^2)\text{ m}}{\text{J}}\right]$$
$$= 1.74 \times 10^{-27} \text{ kg m/s}$$

b. The momentum of the second photon is found similarly

$$p = \frac{h}{\lambda} = \frac{6.625 \times 10^{-34} \text{ J s}}{720.0 \text{ nm}}$$
$$= 9.20 \times 10^{-28} \text{ kg m/s}$$

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According to this quantum theory of light, light spreads out from a source in small bundles of energy called quanta or photons. *Although the photon is treated as a particle, its properties of mass, energy, and momentum are described in terms of frequency or wavelength, strictly a wave concept.*

Thus, we say that *light has a dual nature. It can act as a wave or it can act as a particle, but never both at the same time.* To answer the question posed at the beginning of this section, is light a wave or a particle, the answer is that light is both a wave and a particle. This dual nature of light is stated in the ***principle of complementarity:*** *The wave theory of light and the quantum theory of light complement each other. In a specific event, light exhibits either a wave nature or a particle nature, but never both at the same time.*

When the wavelength of an electromagnetic wave is long, its frequency and hence its photon energy ($E = h\nu$) are small and we are usually concerned with the wave characteristics of the electromagnetic wave. For example, radio and television waves have relatively long wavelengths and they are usually treated as waves. When the wavelength of the electromagnetic wave is small, its frequency and hence its photon energy are large. The electromagnetic wave is then usually considered as a particle. For example, X rays have very small wavelengths and are usually treated as particles. However, this does not mean that X rays cannot also act as waves. In fact they do. When X rays are scattered from a crystal, they behave like waves, exhibiting the usual diffraction patterns associated with waves. The important thing is that light can act either as a wave or a particle, but never both at the same time.

Let us *summarize the characteristics of the photon:*

Rest Mass $$m_0 = 0 \tag{3.14}$$

Energy $$E = h\nu \tag{3.6}$$

Mass $$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \tag{3.16}$$

Momentum of Photon $$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \tag{3.21}$$

Although the two examples considered were for photons of visible light, do not forget that the photon is a particle in the entire electromagnetic spectrum.

Example 3.6

The mass of an X ray and a gamma ray. Find the mass of a photon for (a) an X ray of 100.0-nm wavelength and (b) for a gamma ray of 0.0500 nm.

Solution

a. The mass of an X-ray photon, found from equation 3.16, is

$$m = \frac{h\nu}{c^2} = \frac{h}{c\lambda}$$

$$= \frac{(6.625 \times 10^{-34}\ \text{J s})}{(3.00 \times 10^{8}\ \text{m/s})(100.0\ \text{nm})}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$

$$= 2.20 \times 10^{-35}\ \text{kg}$$

b. The mass of the gamma ray is

$$m = \frac{h\nu}{c^2} = \frac{h}{c\lambda}$$

$$= \frac{(6.625 \times 10^{-34}\ \text{J s})}{(3.00 \times 10^{8}\ \text{m/s})(0.0500\ \text{nm})}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$

$$= 4.42 \times 10^{-32}\ \text{kg}$$

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Comparing the mass of a photon for red light, violet light, X rays, and gamma rays we see

$$m_{\textbf{red}} = 3.07 \times 10^{-36}\ \text{kg}$$
$$m_{\text{violet}} = 5.81 \times 10^{-36}\ \text{kg}$$
$$m_{\text{X ray}} = 22.0 \times 10^{-36}\ \text{kg}$$
$$m_{\text{gamma ray}} = 44{,}200 \times 10^{-36}\ \text{kg}$$

Thus, as the frequency of the electromagnetic spectrum increases (wavelength decreases), the mass of the photon increases.

3.5 The Compton Effect

If light sometimes behaves like a particle, the photon, why not consider the collision of a photon with a free electron from the same point of view as the collision of two billiard balls? Such a collision between a photon and a free electron is called Compton scattering, or the **Compton effect,** in honor of Arthur Holly Compton (1892-1962). In order to get a massive photon for the collision, X rays are used. (Recall that X rays have a high frequency ν, and therefore the energy of the X ray, $E = h\nu$, is large, and thus its mass, $m = E/c^2$, is also large.) In order to get a free electron, a target made of carbon is used. The outer electrons of the carbon atom are very loosely bound, so compared with the initial energy of the photon, the electron looks like a free electron. Thus, the collision between the photon and the electron can be pictured as shown in figure 3.7. We assume that the electron is initially at

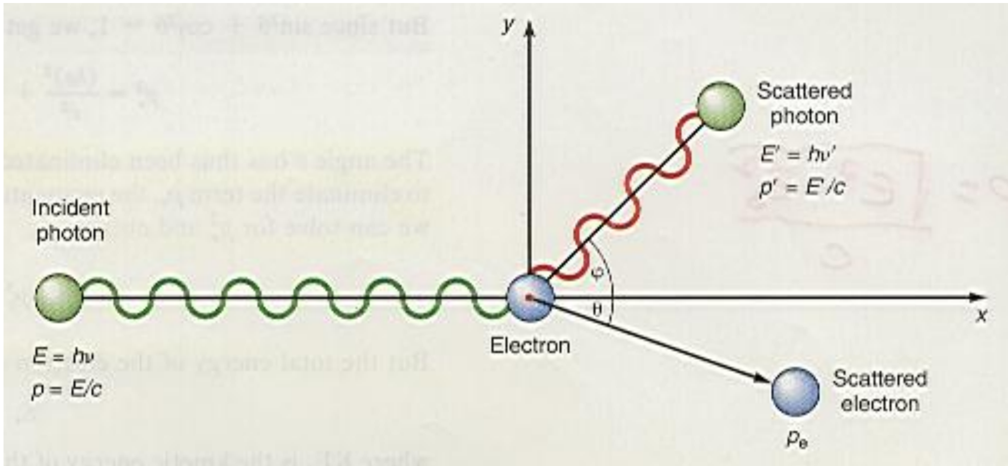

Figure 3.7 Compton scattering.

rest and that the incident photon has an energy ($E = h\nu$) and a momentum ($p = E/c$). After the collision, the electron is found to be scattered at an angle θ from the original direction of the photon. *Because the electron has moved after the collision, some energy must have been imparted to it. But where could this energy come from? It must have come from the incident photon. But if that is true, then the scattered photon must have less energy than the incident photon, and therefore, its wavelength should also have changed.* Let us call the energy of the scattered photon E', where

$$E' = h\nu'$$

and hence, its final momentum is

$$p' = \frac{E'}{c} = \frac{h\nu'}{c}$$

Because momentum is conserved in all collisions, the law of conservation of momentum is applied to the collision of figure 3.7. First however, notice that the collision is two dimensional. Because the vector momentum is conserved, the x-component of the momentum and the y-component of momentum must also be conserved. The law of conservation for the x-component of momentum can be written as

$$p_p + 0 = p_p' \cos\phi + p_e \cos\theta$$

and for the y-component,

$$0 + 0 = p_p' \sin\phi - p_e \sin\theta$$

where p_p is the momentum of the incident photon, p_p' the momentum of the scattered photon, and p_e the momentum of the scattered electron. Substituting the values for the energy and momentum of the photon, these equations become

$$\frac{h\nu}{c} = \frac{h\nu'}{c} \cos\phi + p_e \cos\theta$$
$$0 = \frac{h\nu'}{c} \sin\phi - p_e \sin\theta$$

There are more unknowns (ν', θ, ϕ, p_e) than we can handle at this moment, so let us eliminate θ from these two equations by rearranging, squaring, and adding them. That is,

$$p_e \cos\theta = \frac{h\nu}{c} - \frac{h\nu'}{c} \cos\phi$$
$$p_e \sin\theta = \frac{h\nu'}{c} \sin\phi$$
$$p_e^2 \cos^2\theta = \frac{(h\nu)^2}{c^2} - \frac{2h\nu h\nu' \cos\phi}{c^2} + \frac{(h\nu')^2}{c^2} \cos^2\phi$$
$$p_e^2 \sin^2\theta = \frac{(h\nu')^2}{c^2} \sin^2\phi$$
$$p_e^2 (\sin^2\theta + \cos^2\theta) = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} (\sin^2\phi + \cos^2\phi) - \frac{2(h\nu)(h\nu')}{c^2} \cos\phi$$

But since $\sin^2\theta + \cos^2\theta = 1$, we get

$$p_e^2 = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2(h\nu)(h\nu')}{c^2} \cos\phi \tag{3.22}$$

The angle θ has thus been eliminated from the equation. Let us now look for a way to eliminate the term p_e, the momentum of the electron. If we square equation 3.19, we can solve for p_e^2 and obtain

$$p_e^2 = \frac{E_e^{\,2} - E_{0e}^2}{c^2} \tag{3.23}$$

But the total energy of the electron E_e, given by equation 1.102, is

$$E_e = \mathrm{KE}_e + E_{0e}$$

where KE_e is the kinetic energy of the electron and E_{0e} is its rest mass. Substituting equation 1.102 back into equation 3.23, gives, for the momentum of the electron,

$$p_e^2 = \frac{(\mathrm{KE}_e + E_{0e})^2 - E_{0e}^2}{c^2}$$
$$= \frac{\mathrm{KE}_e^2 + 2E_{0e}\,\mathrm{KE}_e + E_0^2 - E_0^2}{c^2}$$
$$p_e^2 = \frac{\mathrm{KE}_e^2 + 2E_{0e}\,\mathrm{KE}_e}{c^2} \tag{3.24}$$

But if the law of conservation of energy is applied to the collision of figure 3.7, we get

$$E = E' + \mathrm{KE}_{\mathbf{e}}$$
$$h\nu = h\nu' + \mathrm{KE}_{\mathbf{e}} \tag{3.25}$$

where E is the total energy of the system, E' is the energy of the scattered photon and $\mathrm{KE}_{\mathbf{e}}$ is the kinetic energy imparted to the electron during the collision. Thus, the kinetic energy of the electron, found from equation 3.25, is

$$\mathrm{KE}_{\mathbf{e}} = h\nu - h\nu' \tag{3.26}$$

Substituting the value of the kinetic energy from equation 3.26 and $E_{0\mathbf{e}} = m_0c^2$, the rest energy of the electron, back into equation 3.24, we get, for the momentum of the electron,

$$p_e^2 = \frac{(h\nu - h\nu')^2 + 2m_0c^2(h\nu - h\nu')}{c^2}$$
$$p_e^2 = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2h\nu h\nu'}{c^2} + 2m_0(h\nu - h\nu') \tag{3.27}$$

Since we now have two separate equations for the momentum of the electron, equations 3.22 and 3.27, we can equate them to eliminate $p_{\mathbf{e}}$. Therefore,

$$\frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2h\nu h\nu'}{c^2} + 2m_0(h\nu - h\nu') = \frac{(h\nu)^2}{c^2} + \frac{(h\nu')^2}{c^2} - \frac{2(h\nu)(h\nu')}{c^2}\cos\phi$$

Simplifying,

$$2m_0(h\nu - h\nu') = \frac{2h\nu h\nu'}{c^2} - \frac{2(h\nu)(h\nu')}{c^2}\cos\phi$$
$$h\nu - h\nu' = \frac{h\nu h\nu'}{m_0c^2}(1 - \cos\phi)$$
$$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0c^2}(1 - \cos\phi)$$

However, since $\nu = c/\lambda$ this becomes

$$\frac{c/\lambda - c/\lambda'}{(c/\lambda)(c/\lambda')} = \frac{h}{m_0c^2}(1 - \cos\phi)$$
$$\lambda\lambda'\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{h}{m_0c}(1 - \cos\phi)$$
$$(\lambda' - \lambda) = \frac{h}{m_0c}(1 - \cos\phi) \tag{3.28}$$

Equation 3.28 is called the Compton scattering formula. It gives the change in wavelength of the scattered photon as a function of the scattering angle ϕ. The quantity,

$$\frac{h}{m_0 c} = 2.426 \times 10^{-12}\ \text{m} = 0.002426\ \text{nm}$$

which has the dimensions of a length, is called the *Compton wavelength*.

Thus, in a collision between an energetic photon and an electron, the scattered light shows a different wavelength than the wavelength of the incident light. In 1923, A. H. Compton confirmed the modified wavelength of the scattered photon and received the Nobel Prize in 1927 for his work.

Example 3.7

Compton scattering. A 90.0-KeV X-ray photon is fired at a carbon target and Compton scattering occurs. Find the wavelength of the incident photon and the wavelength of the scattered photon for scattering angles of (a) 30.0^0 and (b) 60.0^0.

Solution

The frequency of the incident photon is found from $E = h\nu$ as

$$\nu = \frac{E}{h} = \left(\frac{90.0\times10^{3}\ \text{eV}}{6.625\times10^{-34}\ \text{J s}}\right)\left(\frac{1.60\times10^{-19}\ \text{J s}}{1\ \text{eV}}\right)$$
$$= 2.17 \times 10^{19}\ \text{Hz}$$

The wavelength of the incident photon is found from

$$\lambda = \frac{c}{\nu} = \left(\frac{3.00\times10^{8}\ \text{m/s}}{2.17\times10^{19}\ \text{1/s}}\right)\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$=0.0138\ \text{nm}$$

The modified wavelength is found from the Compton scattering formula, equation 3.28, as

$$\lambda' = \lambda + \frac{h}{m_0 c}(1 - \cos\phi)$$

a.

$$\lambda' = 0.0138\ \text{nm} + (0.002426\ \text{nm})(1 - \cos 30.0^0)$$
$$= 0.0141\ \text{nm}$$

b.

$$\lambda' = 0.0138\ \text{nm} + (0.002426\ \text{nm})(1 - \cos 60.0^0)$$
$$= 0.0150\ \text{nm}$$

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In an actual experiment both the incident and modified wavelengths are found in the scattered photons. The incident wavelength is found in the scattered photons because some of the incident photons are scattered by the atom. In this case, the rest mass of the electron m_0 must be replaced in equation 3.28 by the mass M of the entire atom. Because M is so much greater than m_0, the Compton wavelength h/MC is so small that the change in wavelength for these photons is too small to be observed. Thus, these incident photons are scattered with the same wavelength.

3.6 The Wave Nature of Particles

We have seen that light displays a dual nature; it acts as a wave and it acts as a particle. Assuming symmetry in nature, the French physicist Louis de Broglie (1892-1987) proposed, in his 1924 doctoral dissertation, that particles should also possess a wave characteristic. Because the momentum of a photon was shown to be

$$p = \frac{h}{\lambda} \tag{3.21}$$

de Broglie assumed that the wavelength of the wave associated with a particle of momentum p, should be given by

$$\lambda = \frac{h}{p} \tag{3.29}$$

Equation 3.29 is called the ***de Broglie relation.*** *Thus, de Broglie assumed that the same wave-particle duality associated with electromagnetic waves should also apply to particles.* Hence, an electron can be considered to be a particle and it can also be considered to be a wave. Instead of solving the problem of the wave-particle duality of electromagnetic waves, de Broglie extended it to include matter as well.

Example 3.8

The wavelength of a particle. Calculate the wavelength of (a) a 0.140-kg baseball moving at a speed of 44.0 m/s, (b) a proton moving at the same speed, and (c) an electron moving at the same speed.

Solution

a. A baseball has an associated wavelength given by equation 3.29 as

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.625 \times 10^{-34}\ \text{J s}}{(0.140\ \text{kg})(44.0\ \text{m/s})}$$
$$= 1.08 \times 10^{-34}\ \text{m}$$

Such a small wavelength cannot be measured and therefore baseballs always appear as particles.

b. The wavelength of the proton, found from equation 3.29, is

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$
$$= \frac{(6.625\times10^{-34}\ \text{J s})}{(1.67\times10^{-27}\ \text{kg})(44.0\ \text{m/s})}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 9.02\ \text{nm}$$

Although this wavelength is small (it is in the X-ray region of the electromagnetic spectrum), it can be detected.

c. The wavelength of the electron is found from

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$
$$= \frac{(6.625\times10^{-34}\ \text{J s})}{(9.11\times10^{-31}\ \text{kg})(44.0\ \text{m/s})}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$
$$= 1.65 \times 10^{4}\ \text{nm}$$

which is a very large wavelength and can be easily detected.

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Note from example 3.8, that because Planck's constant h is so small, the wave nature of a particle does not manifest itself unless the mass m of the particle is also very small (of the order of an atom or smaller). This is why the wave nature of particles is not part of our everyday experience.

de Broglie's hypothesis was almost immediately confirmed when in 1927 C. J. Davisson and L. H. Germer performed an experiment that showed that electrons could be diffracted by a crystal. G. P. Thomson performed an independent experiment at the same time by scattering electrons from very thin metal foils and obtained the standard diffraction patterns that are usually associated with waves. Since that time diffraction patterns have been observed with protons, neutrons, hydrogen atoms, and helium atoms, thereby giving substantial evidence for the wave nature of particles.

For his work on the dual nature of particles, de Broglie received the 1929 Nobel Prize in physics. Davisson and Thomson shared the Nobel Prize in 1937 for their experimental confirmation of the wave nature of particles.

3.7 The Wave Representation of a Particle

We have just seen that a particle can be represented by a wave. The wave associated with a photon was an electromagnetic wave. *But what kind of wave is associated with a particle?* It is certainly not an electromagnetic wave. de Broglie called the wave a *pilot wave* because he believed that it steered the particle during its motion. The waves have also been called *matter waves* to show that they are associated with matter. Today, the wave is simply referred to as the *wave function* and is represented by Ψ.

Because this wave function refers to the motion of a particle we say that *the value of the wave function Ψ is related to the probability of finding the particle at a specific place and time.* The probability P that something can be somewhere at a certain time, can have any value between 0 and 1. If the probability $P = 0$, then there is an absolute certainty that the particle is absent. If the probability $P = 1$, then there is an absolute certainty that the particle is present. If the probability P lies somewhere between 0 and 1, then that value is the probability of finding the particle there. That is, if the probability $P = 0.20$, there is a 20% probability of finding the particle at the specified place and time.

Because the amplitude of any wave varies between positive and negative values, the wave function Ψ cannot by itself represent the probability of finding the particle at a particular time and place. However, the quantity Ψ^2 is always positive and is called the probability density. *The probability density Ψ^2 is the probability of finding the particle at the position (x, y, z) at the time t.* The new science of wave mechanics, or as it was eventually called, quantum mechanics, has to do with determining the wave function Ψ for any particle or system of particles.

How can a particle be represented by a wave? Recall from general physics, that a wave moving to the right is defined by the function

$$y = A\sin(kx - \omega t)$$

where the wave number k is

$$k = \frac{2\pi}{\lambda}$$

and the angular frequency ω is given by

$$\omega = 2\pi f$$

or since $f = \nu$, in our new notation,

$$\omega = 2\pi\nu$$

Also recall that the velocity of the wave is given by

$$v = \frac{\omega}{k}$$

We will therefore begin, in our analysis of matter waves, by trying to define the wave function as

$$\Psi = A \sin(kx - \omega t) \tag{3.30}$$

A plot of this wave function for $t = 0$ is shown in figure 3.8(a). The first thing to observe in this picture is that the wave is too spread out to be able to represent a particle. Remember the particle must be found somewhere within the wave. Because the wave extends out to infinity the particle could be anywhere.

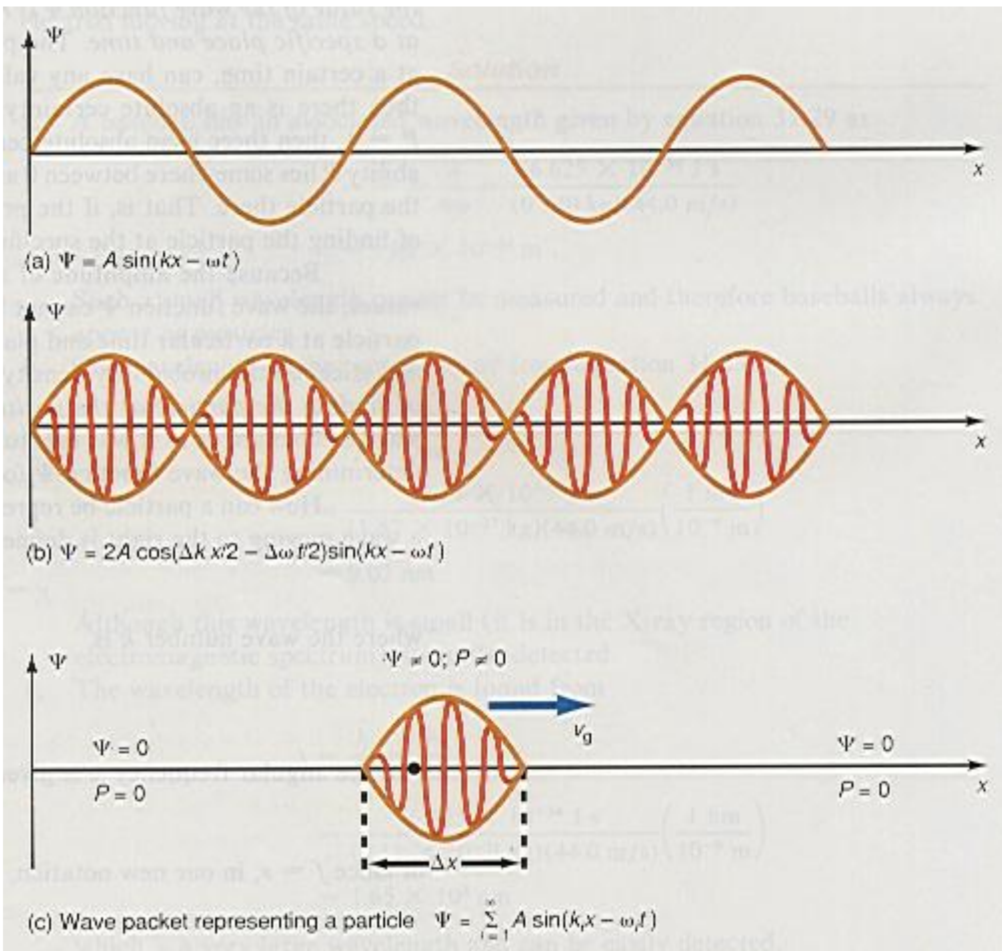

Figure 3.8 Representation of a particle as a wave.

Because one of the characteristics of waves is that they obey the superposition principle, perhaps a wave representation can be found by adding different waves together. As an example, let us add two waves of slightly different wave numbers and slightly different angular frequencies. That is, consider the two waves

$$\Psi_1 = A\sin(k_1x - \omega_1 t)$$
$$\Psi_2 = A\sin(k_2x - \omega_2 t)$$

where

$$k_2 = k_1 + \Delta k$$

and

$$\omega_2 = \omega_1 + \Delta\omega$$

The addition of these two waves gives

$$\Psi = \Psi_1 + \Psi_2$$
$$= A\sin(k_1x - \omega_1 t) + A\sin(k_2x - \omega_2 t)$$

The addition of two sine waves is shown in appendix B, to be

$$\sin B + \sin C = 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

Letting

$$B = k_1x - \omega_1 t$$

and

$$C = k_2x - \omega_2 t$$

we find

$$\Psi = 2A\sin\left(\frac{k_1x - \omega_1 t + k_2x - \omega_2 t}{2}\right)\cos\left(\frac{k_1x - \omega_1 t - k_2x + \omega_2 t}{2}\right)$$
$$= 2A\sin\left[\frac{k_1x - \omega_1 t + (k_1 + \Delta k)x - (\omega_1 + \Delta\omega)t}{2}\right]\cos\left(\frac{k_1x - \omega_1 t - (k_1 + \Delta k)x + (\omega_1 + \Delta\omega)t}{2}\right)$$
$$= 2A\sin\left[\frac{2kx + (\Delta k)x - 2\omega t - (\Delta\omega)t}{2}\right]\cos\left(\frac{-\Delta k}{2}x + \frac{\Delta\omega}{2}t\right)$$

We have dropped the subscript 1 on k and ω to establish the general case. Now as an approximation

$$2kx + (\Delta k)x \approx 2kx$$

and

$$-2\omega t - (\Delta\omega)t \approx -2\omega t$$

Therefore,

$$\Psi = 2A\sin(kx - \omega t)\cos\left(\frac{-\Delta k}{2}x + \frac{\Delta\omega}{2}t\right)$$

One of the properties of the cosine function is that $\cos(-\theta) = \cos\theta$. Using this relation the wave function becomes

$$\Psi = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\sin(kx - \omega t) \tag{3.31}$$

A plot of equation 3.31 is shown in figure 3.8(b). The amplitude of this wave is modulated and is given by the first part of equation 3.31 as

$$A_m = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \tag{3.32}$$

This wave superposition gives us a closer representation of a particle. Each modulated portion of the wave represents a group of waves and any one group can represent a particle. The velocity of the group of waves represents the velocity of the particle.

Equation 3.31 and figure 3.8(b) approaches a wave representation of the particle. If an infinite number of waves, each differing slightly in wave number and angular frequency, were added together we would get the wave function

$$\Psi = \sum_{i=1}^{\infty} A\sin(k_i x - \omega_i t) \tag{3.33}$$

which is shown in figure 3.8(c) and is called a *wave packet. This wave packet can indeed represent the motion of a particle. Because the wave function* Ψ *is zero everywhere except within the packet, the probability of finding the particle is zero everywhere except within the packet.* The wave packet localizes the particle to be within the region Δx shown in figure 3.8(c), and the wave packet moves with the group velocity of the waves and this is the velocity of the particle. The fundamental object of wave mechanics or quantum mechanics is to find the wave function Ψ associated with a particle or a system of particles.

3.8 The Heisenberg Uncertainty Principle

One of the characteristics of the dual nature of matter is a fundamental limitation in the accuracy of the measurement of the position and momentum of a particle. This can be seen in a very simplified way by looking at the modulated wave of figure 3.8(b) and reproduced in figure 3.9. A particle is shown located in the first group of the modulated wave. Since the particle lies somewhere within the wave packet its exact position is uncertain. The amount of the uncertainty in its position is no greater than Δx, the width of the entire wave packet or wave group. The wavelength of the modulated amplitude λ_m is shown in figure 3.9 and we can see that a wave group is only half that distance. Thus, the uncertainty in the location of the particle is given by

$$\Delta x = \frac{\lambda_m}{2} \tag{3.34}$$

The uncertainty in the momentum can be found by solving the de Broglie relation, equation 3.29, for momentum as

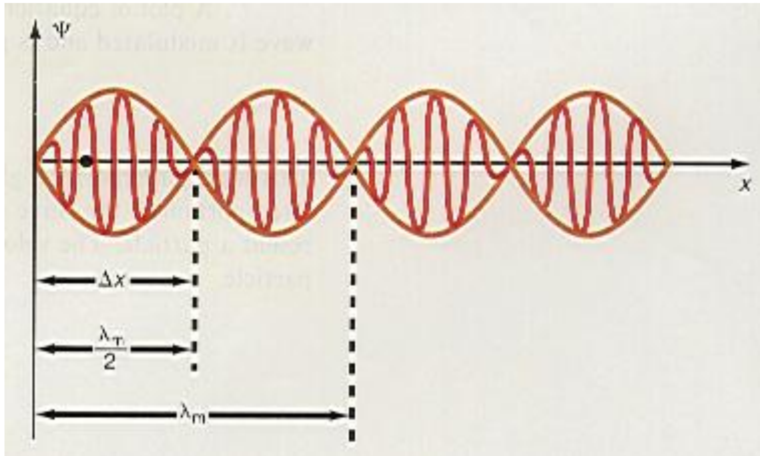

Figure 3.9 Limitations on position and momentum of a particle.

$$p = \frac{h}{\lambda} \tag{3.35}$$

and the fact that the wavelength is given in terms of the wave number by

$$\lambda = \frac{2\pi}{k} \tag{3.36}$$

Substituting equation 3.36 into equation 3.35 gives

$$p = \frac{h}{\lambda} = \frac{h}{2\pi/k} = \frac{h}{2\pi}k \tag{3.37}$$

The uncertainty in the momentum, found from equation 3.37, is

$$\Delta p = \frac{h}{2\pi}\Delta k \tag{3.38}$$

Because the wave packet is made up of many waves, there is a Δk associated with it. This means that in representing a particle as a wave, there is automatically an uncertainty in the wave number, k, which we now see implies an uncertainty in the momentum of that particle. For the special case considered in figure 3.9, the wave number of the modulated wave Δk_m is found from

$$A_m = 2A\cos(k_m x - \omega_m t)$$

and from equation 3.32 as

$$k_m = \frac{\Delta k}{2} \tag{3.39}$$

But from the definition of a wave number

$$k_m = \frac{2\pi}{\lambda_m} \tag{3.40}$$

Substituting equation 3.40 into equation 3.39 gives, for Δk,

$$\Delta k = 2k_m = 2\left(\frac{2\pi}{\lambda_m}\right) \tag{3.41}$$

Substituting the uncertainty for Δk, equation 3.41, into the uncertainty for Δp, equation 3.38, gives

$$\Delta p = \frac{h}{2\pi}\Delta k = \frac{h}{2\pi}2\left(\frac{2\pi}{\lambda_m}\right) = \frac{h}{\lambda_m / 2} \tag{3.42}$$

The uncertainty between the position and momentum of the particle is obtained by substituting equation 3.34 for $\lambda_m/2$ into equation 3.42 to get

$$\Delta p = \frac{h}{\Delta x}$$

or

$$\Delta p \Delta x = h \tag{3.43}$$

Because Δp and Δx are the smallest uncertainties that a particle can have, their values are usually greater than this, so their product is usually greater than the value of h. To show this, equation 3.43 is usually written with an inequality sign also, that is,

$$\Delta p \Delta x \geq h$$

The analysis of the wave packet was greatly simplified by using the modulated wave of figure 3.8(b). A more sophisticated analysis applied to the more reasonable wave packet of figure 3.8(c) yields the relation

$$\Delta p \Delta x \geq \hbar \tag{3.44}$$

where the symbol $\hbar$, called h bar, is

$$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34} \text{ J s} \tag{3.45}$$

Equation 3.44 is called the ***Heisenberg uncertainty principle.*** *It says that the position and momentum of a particle cannot both be measured simultaneously with perfect accuracy. There is always a fundamental uncertainty associated with any measurement. This uncertainty is not associated with the measuring instrument. It is a consequence of the wave-particle duality of matter.*

As an example of the application of equation 3.44, if the position of a particle is known exactly, then $\Delta x = 0$ and Δp would have to be infinite in order for the product $\Delta x \Delta p$ to be greater than $\hbar$. If Δp is infinite, the value of the momentum of the particle is completely unknown. A wave packet associated with a very accurate value of position is shown in figure 3.10(a). Although this wave packet

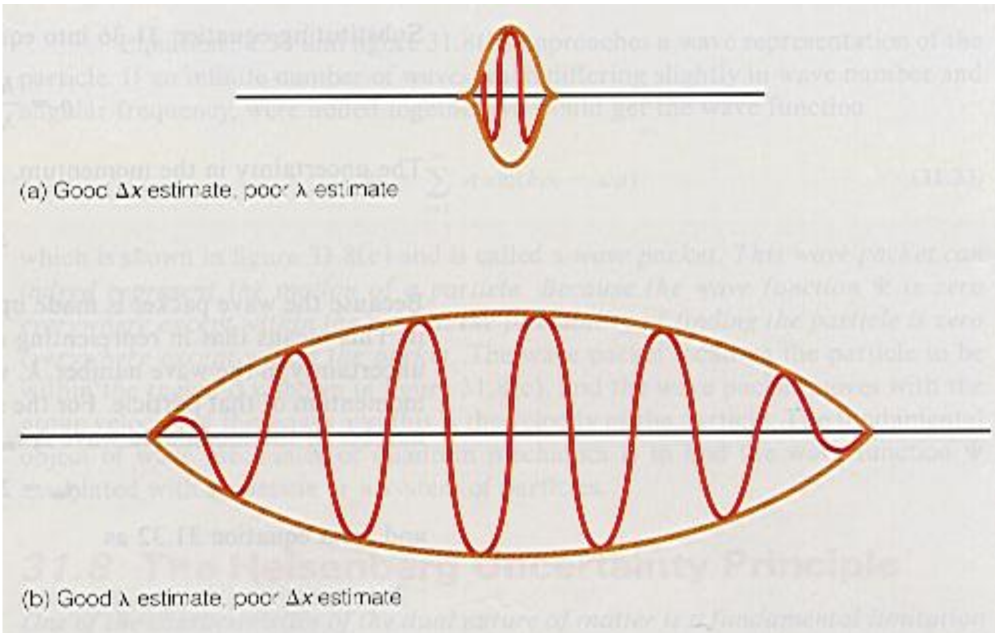

Figure 3.10 Wave packets of different size.

gives a very small value of Δx, it gives an exceedingly poor representation of the wavelength. Because the uncertainty in the wavelength λ is large, the uncertainty in the wave number is also large. Since the uncertainty in the wave number is related to the uncertainty in the momentum of the particle by equation 3.38, there is also a large uncertainty in the momentum of the particle. Thus, a good Δx estimate always gives a poor Δp estimate.

If the momentum of a particle is known exactly, then $\Delta p = 0$, and this implies that Δx must approach infinity. That is, if the momentum of a particle is known exactly, the particle could be located anywhere. A wave packet approximating this case is shown in figure 3.10(b). Because the wave packet is spread out over a large area it is easy to get a good estimate of the de Broglie wavelength, and hence a good estimate of the momentum of the particle. On the other hand, since the wave packet is so spread out, it is very difficult to locate the particle inside the wave packet. Thus a good Δp estimate always gives a poor Δx estimate.

Example 3.9

The uncertainty in the velocity of a baseball. A 0.140-kg baseball is moving along the x-axis. At a particular instant of time it is located at the position $x = 0.500$ m with an uncertainty in the measurement of Δx = 0.001 m. How accurately can the velocity of the baseball be determined?

Solution

The uncertainty in the momentum is found by the Heisenberg uncertainty principle, equation 3.44, as

$$\Delta p \geq \frac{\hbar}{\Delta x}$$
$$\geq \frac{1.05 \times 10^{-34} \text{ J s}}{0.001 \text{ m}}$$
$$\geq 1.05 \times 10^{-31} \text{ kg m/s}$$

Since $p = mv$, the uncertainty in the velocity is

$$\Delta v \geq \frac{\Delta p}{m}$$
$$= \frac{1.05 \times 10^{-31} \text{ kg m/s}}{0.140 \text{ kg}}$$
$$\geq 7.50 \times 10^{-31} \text{ m/s}$$

The error in Δp and Δv caused by the uncertainty principle is so small for macroscopic bodies moving around in the everyday world that it can be neglected.

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Example 3.10

The uncertainty in the velocity of an electron confined to a box the size of the nucleus. We want to confine an electron, $m_e = 9.11 \times 10^{-31}$ kg, to a box, 1.00×10^{-14} m long (approximately the size of a nucleus). What would the speed of the electron be if it were so confined?

Solution

Because the electron can be located anywhere within the box, the worst case of locating the electron is for the uncertainty of the location of the electron to be equal to the size of the box itself. That is, $\Delta x = 1.00 \times 10^{-14}$ m.

We also assume that the uncertainty in the velocity is so bad that it is equal to the velocity of the electron itself. The uncertainty in the speed, found from the Heisenberg uncertainty principle, is

$$\Delta p \geq \frac{\hbar}{\Delta x}$$

$$m\Delta v \geq \frac{\hbar}{\Delta x}$$

$$\Delta v \geq \frac{\hbar}{m\Delta x} \tag{3.46}$$

$$\geq \frac{1.05\times10^{-34}\ \text{J s}}{(9.11\times10^{-31}\ \text{kg})(1.00\times10^{-14}\ \text{m})}\left[\frac{(\text{kg m/s}^2)\ \text{m}}{\text{J}}\right]$$

$$\geq 1.15\times10^{10}\ \text{m/s}$$

Hence, for the electron to be confined in a box about the size of the nucleus, its speed would have to be greater than the speed of light. Because this is impossible, we must conclude that an electron can never be found inside of a nucleus.

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Example 3.11

The uncertainty in the velocity of an electron confined to a box the size of an atom. An electron is placed in a box about the size of an atom, that is, $\Delta x = 1.00 \times 10^{-12}$ m. What is the velocity of the electron?

Solution

We again assume that the velocity of the electron is of the same order as the uncertainty in the velocity, then from equation 3.46, we have

$$\Delta v \geq \frac{\hbar}{m\Delta x}$$

$$\geq \frac{1.05\times10^{-34}\ \text{J s}}{(9.11\times10^{-31}\ \text{kg})(1.00\times10^{-12}\ \text{m})}$$

$$\geq 1.15\times10^{8}\ \text{m/s}$$

Because this velocity is less than the velocity of light, an electron can exist in an atom. Notice from these examples that the uncertainty principle is only important on the microscopic level.

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Another way to observe the effect of the uncertainty principle from a more physical viewpoint is to see what happens when we "see" a particle in order to locate its position. Figure 3.11(a) shows how we locate a moving baseball. The

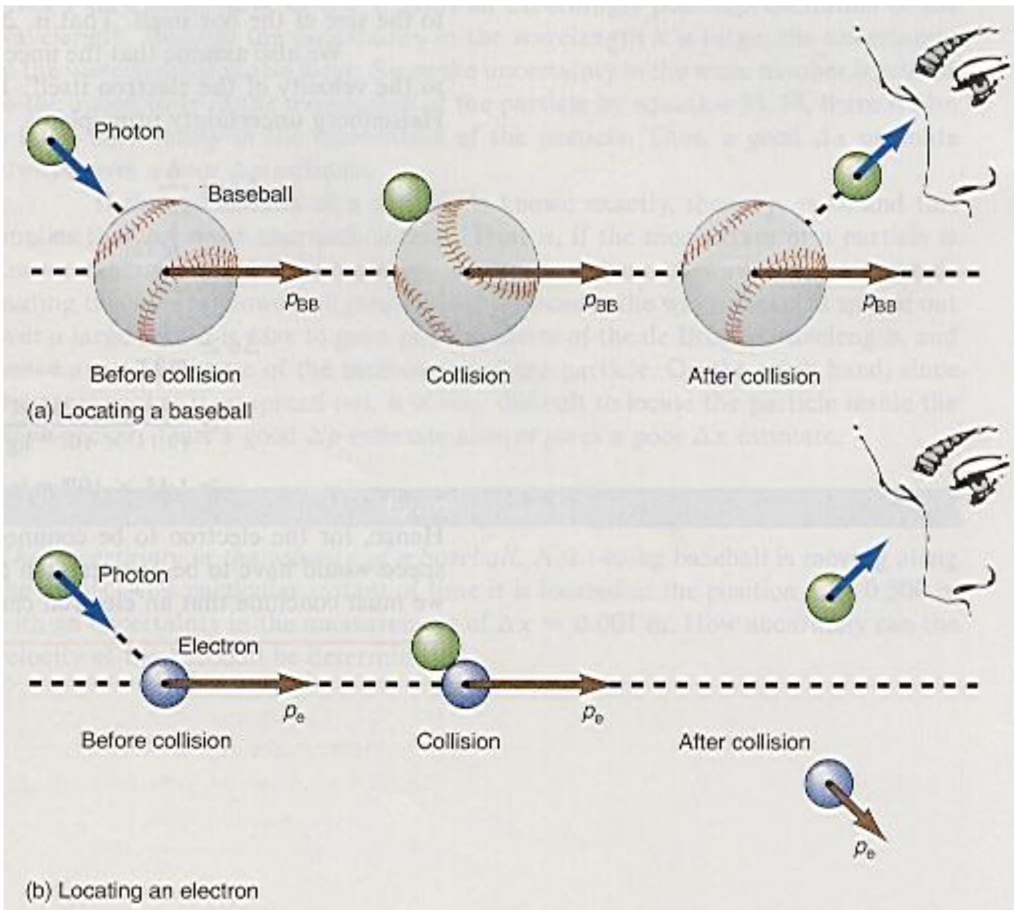

Figure 3.11 The Heisenberg uncertainty principle.

process is basically a collision between the photon of light and the baseball. The photon hits the baseball and then bounces off (is reflected) and proceeds to our eye. We then can say that we saw the baseball at a particular location. Because the mass of the photon is so small compared to the mass of the baseball, the photon bounces off the baseball without disturbing the momentum of the baseball. Thus, in the process of "locating" the baseball, we have done nothing to disturb its momentum.

Now let us look at the problem of "seeing" an electron, figure 3.11(b). The process of "seeing" again implies a collision between the photon of light and the object we wish to see; in this case, the electron. However, the momentum of the photon is now of the same order of magnitude as the momentum of the electron. Hence, as the photon hits the electron, the electron's momentum is changed just as in the Compton effect. Thus, we have located the electron by "seeing" it, but in the process of "seeing" it, we have disturbed or changed its momentum. *Hence, in the process of determining its position, we have caused an uncertainty in its momentum. The uncertainty occurs because the mass of the photon is of the same order of magnitude as the mass of the electron. Thus, the uncertainty always occurs when dealing with microscopic objects.*

The classical picture of being able to predict the exact position and velocity of a particle by Newton's second law and the kinematic equations obviously does not hold in the microscopic region of atoms because of the uncertainty principle. The exact positions and velocities are replaced by a probabilistic determination of position and velocity. That is, we now speak of the probability of finding a particle at a particular position, and the probability that its velocity is a particular value.

On the macroscopic level, the mass of the photon is totally insignificant with respect to the mass of the macroscopic body we wish to see and there is, therefore, no intrinsic uncertainty in measuring the position and velocity of the particle. This is why we are not concerned with the uncertainty principle in classical mechanics.

3.9 Different Forms of the Uncertainty Principle

The limitation on simultaneous measurements is limited not only to the position and momentum of a particle but also to its angular position and angular momentum, and also to its energy and the time in which the measurement of the energy is made.

One of the ways that the angular momentum of a particle is defined is

$$L = rp \sin \theta \tag{3.47}$$

For a particle moving in a circle of radius r, the velocity, and hence the momentum is perpendicular to the radius. Hence, $\theta = 90^0$, and $\sin 90^0 = 1$. Thus, we can also write the angular momentum of a particle as the product of the radius of the circle and the linear momentum of the particle. That is,

$$L = rp \tag{3.48}$$

With this definition of angular momentum, we can easily see the effect of the uncertainty principle on a particle in rotational motion.

Calling x the displacement of a particle along the arc of the circle, when the particle moves through the angle θ, we have

$$x = r\theta$$

The uncertainty Δx in terms of the uncertainty $\Delta\theta$ in angle, becomes

$$\Delta x = r\Delta\theta$$

Substituting this uncertainty into Heisenberg's uncertainty relation, we get

$$\Delta x \Delta p \geq \hbar$$
$$r\Delta\theta\Delta p \geq \hbar$$
$$(\Delta\theta)(r\Delta p) \geq \hbar \tag{3.49}$$

But equation 3.48, which gave us the angular momentum of the particle, also gives us the uncertainty in this angular momentum as

$$\Delta L = r\Delta p \tag{3.50}$$

But this is exactly one of the terms in equation 3.49. Therefore, substituting equation 3.50 into equation 3.49 gives the Heisenberg uncertainty principle for rotational motion as

$$\Delta\theta \, \Delta L \geq \hbar \tag{3.51}$$

Heisenberg's uncertainty principle in this form says that the product of the uncertainty in the angular position and the uncertainty in the angular momentum of the particle is always equal to or greater than the value $\hbar$. Thus, if the angular position of a particle is known exactly, $\Delta\theta = 0$, then the uncertainty in the angular momentum is infinite. On the other hand, if the angular momentum is known exactly, $\Delta L = 0$, then we have no idea where the particle is located in the circle.

The relationship between the uncertainty in the energy of a particle and the uncertainty in the time of its measurement is found as follows. Because the velocity of a particle is given by $v = \Delta x/\Delta t$, the distance that the particle moves during the measurement process is

$$\Delta x = v\Delta t \tag{3.52}$$

The momentum of the particle is given by the de Broglie relation as

$$p = \frac{h}{\lambda} = \frac{h\nu}{v} = \frac{E}{v} \tag{3.53}$$

because $1/\lambda = \nu/v$ and $h\nu = E$. The uncertainty of momentum in terms of the uncertainty in its energy, found from equation 3.53, is

$$\Delta p = \frac{\Delta E}{v} \tag{3.54}$$

Substituting equations 3.52 and 3.54 into the Heisenberg uncertainty relation, gives

$$\Delta x \Delta p \geq \hbar$$

$$(v\Delta t)\left(\frac{\Delta E}{v}\right) \geq \hbar$$

or

$$\Delta E \Delta t \geq \hbar \tag{3.55}$$

Equation 3.55 says that the product of the uncertainty in the measurement of the energy of a particle and the uncertainty in the time of the measurement of the particle is always equal to or greater than $\hbar$. Thus, in order to measure the energy of a particle exactly, $\Delta E = 0$, it would take an infinite time for the measurement. To measure the particle at an exact instant of time, $\Delta t = 0$, we will have no idea of the energy of that particle (ΔE would be infinite).

Example 3.12

The uncertainty in the energy of an electron in an excited state. The lifetime of an electron in an excited state is about 10^{-8} s. (This is the time it takes for the electron to stay in the excited state before it jumps back to the ground state.) What is its uncertainty in energy during this time?

Solution

The energy uncertainty, found from equation 3.55, is

$$\Delta E \Delta t \geq \hbar$$
$$\Delta E \geq \frac{\hbar}{\Delta t}$$
$$\geq \frac{1.05 \times 10^{-34} \text{ J s}}{1.00 \times 10^{-8} \text{ s}}$$
$$\geq 1.05 \times 10^{-26} \text{ J}$$

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3.10 The Heisenberg Uncertainty Principle and Virtual Particles

It is a truly amazing result of the uncertainty principle that it is possible to violate the law of conservation of energy by borrowing an amount of energy ΔE*, just as long as it is paid back before the time* Δt*, required by the uncertainty principle, equation 3.55, has elapsed.* That is, the energy ΔE can be borrowed if it is paid back before the time

$$\Delta t = \frac{\hbar}{\Delta E} \tag{3.56}$$

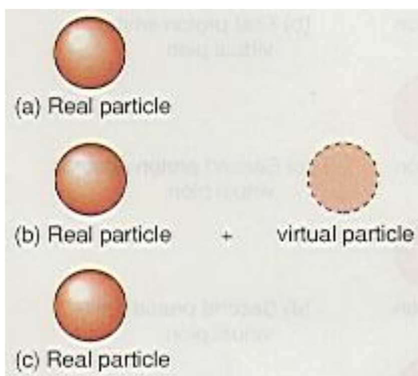
This borrowed energy can be used to create particles. The borrowed energy ΔE is converted to a mass Δm, given by Einstein's mass-energy relation

$$\Delta E = (\Delta m)c^2$$

The payback time thus becomes

$$\Delta t = \frac{\hbar}{(\Delta m)c^2} \tag{3.57}$$

These particles must have very short lifetimes because the energy must be repaid before the elapsed time Δt. These ghostlike particles are called **virtual particles.** Around any real particle there exists a host of these virtual particles. We can visualize virtual particles with the help of figure 3.12. The real particle is shown in figure 3.12(a). In the short period of time Δt, another particle, the virtual particle, materializes as in figure 3.12(b). Before the time Δt is over, the virtual particle returns to the original particle, repaying its energy, and leaving only the real particle, figure 3.12(c). The original particle continues to fluctuate into the two particles.



(a) Real particle (b) Real particle + virtual particle (c) Real particle
Figure 3.12 The virtual particle.

We can determine approximately how far the virtual particle moves away from the real particle by assuming that the maximum speed at which it could possibly move is the speed of light. The distance that the virtual particle can move and then return is then found from

$$d = c\,\frac{\Delta t}{2} \tag{3.58}$$

As an example, suppose the real particle is a proton. Let us assume that we borrow enough energy from the proton to create a particle called the pi-meson (*pion* for short). The mass of the pion is about 2.48×10^{-28} kg. How long can this virtual pion live? From equation 3.57, we have

$$\Delta t = \frac{\hbar}{(\Delta m)c^2}$$

$$= \frac{1.05 \times 10^{-34} \text{ J s}}{(2.48 \times 10^{-28} \text{ kg})(3.00 \times 108 \text{ m/s})^2}$$

$$= 4.70 \times 10^{-24} \text{ s}$$

The approximate distance that the pion can move in this time and return, found from equation 3.58, is

$$d = c\,\frac{\Delta t}{2}$$

$$= (3.00 \times 10^8 \text{ m/s})\frac{(4.70 \times 10^{-24} \text{ s})}{2}$$

$$= 0.705 \times 10^{-15} \text{ m}$$

This distance is, of course, only approximate because the pion does not move at the speed of light. However, the calculation does give us the order of magnitude of the distance. What is interesting is that the radius of the nucleus of hydrogen is 1.41×10^{-15} m and for uranium it is 8.69×10^{-15} m. Thus, the distance that a virtual particle can move is of the order of the size of the nucleus.

If there are two real protons relatively close together as in the nucleus of an atom as shown in figure 3.13(a), then one proton can emit a virtual pion that can travel to the second proton, figure 3.13(b). The second proton can absorb the virtual

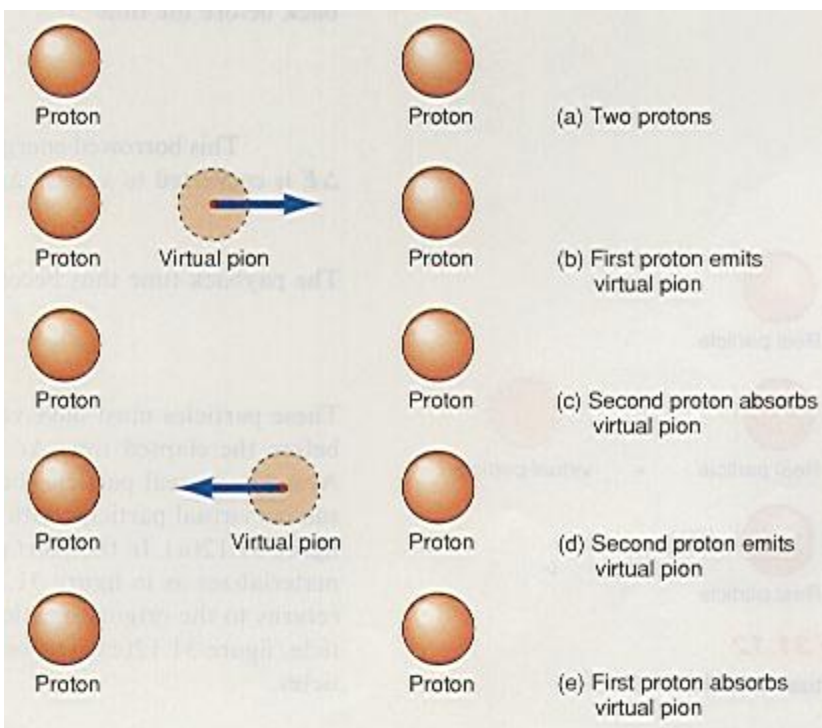


Figure 3.13 The exchange of a virtual pion.

pion, figure 3.13(c). The second pion then emits a virtual pion that can travel to the first pion, figure 3.13(d). The first proton then absorbs the virtual pion, figure 3.13(e). Thus, the protons can exchange virtual pions with one another. *In 1934, the Japanese physicist, Hideki Yukawa (1907-1981), proposed that if two protons exchanged virtual mesons, the result of the exchange would be a very strong attractive force between the protons.* The exchange of virtual mesons between neutrons would also cause a strong attractive force between the neutrons. This exchange force must be a very short-ranged force because it is not observed anywhere outside of the nucleus. The predicted pi-meson was found in cosmic rays by Cecil F. Powell in 1947. Yukawa won the Nobel Prize in physics in 1949, and Powell in 1950.

The concept of a force caused by the exchange of particles is a quantum mechanical concept that is not found in classical physics. The best way to try to describe it classically is to imagine two boys approaching each other on roller skates, as shown in figure 3.14(a). Each boy is moving in a straight line as they approach each other. When the boys are relatively close, the first boy throws a

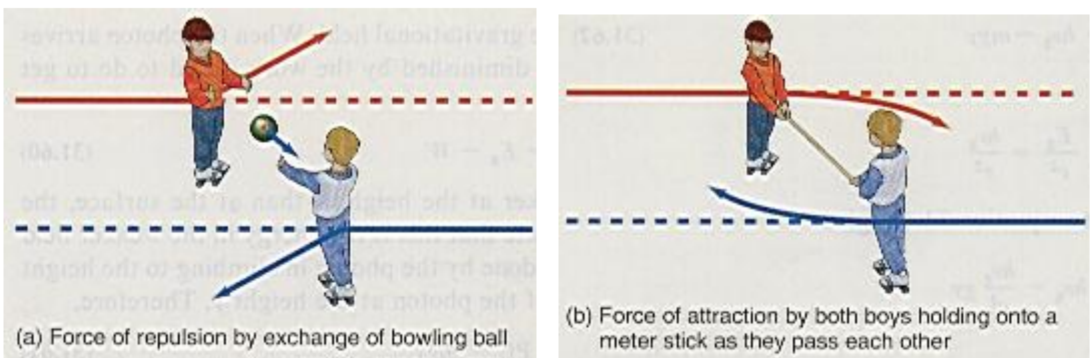


Figure 3.14 Classical analogy of exchange force.

bowling ball to the second boy. By the law of conservation of momentum the first boy recoils after he throws the ball, whereas the second boy recoils after he catches it. The net effect of throwing the bowling ball is to deviate the boys from their straight line motion as though a force of repulsion acted on the two boys. In this way we can say that the exchange of the bowling ball caused a repulsive force between the two boys.

A force of attraction can be similarly analyzed. Suppose, again, that the two boys are approaching each other on roller skates in a straight line motion. When the boys are relatively close the first boy holds out a meterstick for the second boy to grab, figure 3.14(b). As both boys hold on to the meterstick as they pass, they exert a force on each other through the meterstick. The force pulls each boy toward the other boy and deviates the straight line motion into the curved motion toward each other. When the first boy lets go of the meterstick, the attractive force disappears and the boys move in a new straight line motion. Thus, the exchange of the meterstick acted like an attractive force.

The exchange of the virtual pions between the protons in the nucleus cause a very large attractive force that is able to overcome the electrostatic force of repulsion between the protons. The virtual pions can be thought of as a *nuclear glue* that holds the nucleus together. *The tremendous importance of the concept of borrowing energy to form virtual particles, a concept that comes from the Heisenberg uncertainty principle, allows us to think of all forces as being caused by the exchange of virtual particles. Thus, the electrical force can be thought of as caused by the exchange of virtual photons and the gravitational force by the exchange of virtual gravitons (a particle not yet discovered).*

3.11 The Gravitational Red Shift by the Theory of Quanta

The relation for the gravitational red shift was derived in chapter 2 by observing how a clock slows down in a gravitational field. A remarkably simple derivation of this red shift can be obtained by treating light as a particle.

Let an atom at the surface of the earth emit a photon of light of frequency ν_g. This photon has the energy

$$E_g = h\nu_g \tag{3.59}$$

The subscript g is to remind us that this is a photon in the gravitational field. Let us assume that the light source was pointing upward so that the photon travels upward against the gravitational field of the earth until it arrives at a height y above the surface, as shown in figure 3.15. (We have used y for the height instead of h, as used previously, so as not to confuse the height with Planck's constant h.) As the photon rises it must do work against the gravitational field. When the photon arrives at the height y, its energy E_f must be diminished by the work it had to do to get there. Thus

$$E_f = E_g - W \tag{3.60}$$

Because the gravitational field is weaker at the height y than at the surface, the subscript f has been used on E to indicate that this is the energy in the weaker field

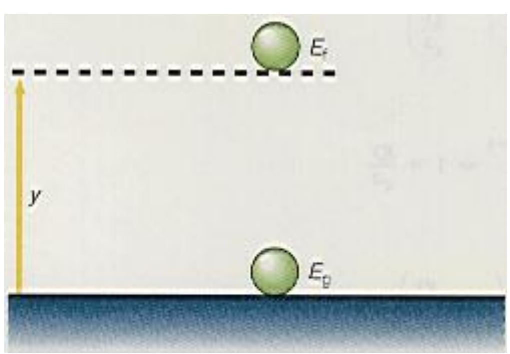


Figure 3.15 A photon in a gravitational field.

or even in a field-free space. The work done by the photon in climbing to the height y is the same as the potential energy of the photon at the height y. Therefore,

$$W = \text{PE} = mgy \tag{3.61}$$

Substituting equation 3.61 and the values of the energies back into equation 3.60, gives

$$h\nu_f = h\nu_g - mgy \tag{3.62}$$

But the mass of the emitted photon is

$$m = \frac{E_g}{c^2} = \frac{h\nu_g}{c^2}$$

Placing this value of the mass back into equation 3.62, gives

$$h\nu_f = h\nu_g - \frac{h\nu_g}{c^2} gy$$

or

$$\nu_f = \nu_g\left(1 - \frac{gy}{c^2}\right) \tag{3.63}$$

Equation 3.63 says that the frequency of a photon associated with a spectral line that is observed away from the gravitational field is less than the frequency of the spectral line emitted by the atom in the gravitational field itself. Since the frequency ν is related to the wavelength λ by $c = \lambda\nu$, the observed wavelength in the field-free space λ_f is longer than the wavelength emitted by the atom in the gravitational field λ_g. Therefore, the observed wavelength is shifted toward the red end of the spectrum. Note the equation 3.63 is the same as equation 2.37. *The slowing down of a clock in a gravitational field follows directly from equation (3.63)* by noting that the frequency ν is related to the period of time T by $\nu = 1/T$. Hence

$$\frac{1}{T_f} = \frac{1}{T_g}\left(1 - \frac{gy}{c^2}\right)$$

$$T_f = \frac{T_g}{1 - gy/c^2}$$

$$T_f = T_g\left(1 - \frac{gy}{c^2}\right)^{-1}$$

But by the binomial theorem,

$$\left(1 - \frac{gy}{c^2}\right)^{-1} = 1 + \frac{gy}{c^2}$$

Thus,

$$T_f = T_g\left(1+\frac{gy}{c^2}\right) \tag{3.64}$$

Equation 3.64 is identical to equation 2.34. Finally calling the period of time T an elapsed time, Δt, we have

$$\Delta t_f = \Delta t_g\left(1+\frac{gy}{c^2}\right) \tag{3.65}$$

which is identical to equation 2.31, which shows the slowing down of a clock in a gravitational field.

3.12 An Accelerated Clock

An extremely interesting consequence of the gravitational red shift can be formulated by invoking Einstein's principle of equivalence discussed in chapter 2. Calling the inertial system containing gravity the K system and the accelerated frame of reference the K' system, Einstein stated, "we assume that we may just as well regard the system K as being in a space free from a gravitational field if we then regard K as uniformly accelerated." *Einstein's principle of equivalence was thus stated as: on a local scale the physical effects of a gravitational field are indistinguishable from the physical effects of an accelerated coordinate system. "Hence the systems K and K ' are equivalent with respect to all physical processes, that is, the laws of nature with respect to K are in entire agreement with those with respect to K '.* " Einstein then *postulated his theory of general relativity, as: The laws of physics are the same in all frames of reference.*

Since a clock slows down in a gravitational field, equation 3.65, using the equivalence principle, an accelerated clock should also slow down. Replacing the acceleration due to gravity g by the acceleration of the clock a, equation 3.65 becomes

$$\Delta t_f = \Delta t_a\left(1+\frac{ay}{c^2}\right) \tag{3.66}$$

Note that the subscript g on Δt_g in equation 3.65 has now been replaced by the subscript a, giving Δt_a, to indicate that this is the time elapsed on the accelerated clock. Notice from equation 3.66 that

$$\Delta t_f > \Delta t_a$$

indicating that time slows down on the accelerated clock. *That is, an accelerated clock runs more slowly than a clock at rest.* In section 1.8 we saw, using the Lorentz transformation equations, that a clock at rest in a moving coordinate system slows down, and called the result the Lorentz time dilation. However, nothing was said at that time to show how the coordinate system attained its velocity. *Except for zero* *velocity, all bodies or reference systems must be accelerated to attain a velocity. Thus, there should be a relation between the Lorentz time dilation and the slowing down of an accelerated clock.* Let us change our notation slightly and call $\Delta t_{\mathbf{f}}$ the time Δt in a stationary coordinate system and $\Delta t_{\mathbf{a}}$ the time interval on a clock that is at rest in a coordinate system that is accelerating to the velocity v. Assuming that the acceleration is constant, we can use the kinematic equation

$$v^2 = v_0{}^2 + 2ay$$

Further assuming that the initial velocity v_0 is equal to zero and solving for the quantity ay we obtain

$$ay = \frac{v^2}{2} \tag{3.67}$$

Substituting equation 3.67 into equation 3.66, yields

$$\Delta t = \Delta t_a \left(1 + \frac{v^2}{2c^2}\right) \tag{3.68}$$

Using the binomial theorem in reverse

$$1 - nx = (1 - x)^{\mathrm{n}}$$

with $x = v^2/c^2$ and $n = -1/2$, we get

$$\left(1 + \frac{v^2}{2c^2}\right) = \left[1 - \left(-\frac{1}{2}\right)\frac{v^2}{c^2}\right] = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.69}$$

Equation 3.68 becomes

$$\Delta t = \frac{\Delta t_a}{\sqrt{1 - v^2/c^2}} \tag{3.70}$$

But this is exactly the time dilation formula, equation 1.64, found by the Lorentz transformation. Thus the Lorentz time dilation is a special case of the slowing down of an accelerated clock. This is a very important result. *Therefore, it is more reasonable to take the slowing down of a clock in a gravitational field, and thus by the principle of equivalence, the slowing down of an accelerated clock as the more basic physical principle. The Lorentz transformation for time dilation can then be derived as a special case of a clock that is accelerated from rest to the velocity v.*

Just as the slowing down of a clock in a gravitational field can be attributed to the warping of spacetime by the mass, it is reasonable to assume that the slowing down of the accelerated clock can also be thought of as the warping of spacetime by the increased mass, due to the increase in the velocity of the accelerating mass.

The Lorentz length contraction can also be derived from this model by the following considerations. Consider the emission of a light wave in a gravitational field. We will designate the wavelength of the emitted light by λ_g, and the period of the light by T_g. The velocity of the light emitted in the gravitational field is given by

$$c_g = \frac{\lambda_g}{T_g} \tag{3.71}$$

We will designate the velocity of light in a region far removed from the gravitational field as c_f for the velocity in a field-free region. The velocity of light in the field-free region is given by

$$c_f = \frac{\lambda_f}{T_f} \tag{3.72}$$

where λ_f is the wavelength of light, and T_f is the period of the light as observed in the field-free region. If the gravitating mass is not too large, then we can make the reasonable assumption that the velocity of light is the same in the gravitational field region and the field-free region, that is, $c_g = c_f$. We can then equate equation 3.71 to equation 3.72 to obtain

$$\frac{\lambda_g}{T_g} = \frac{\lambda_f}{T_f}$$

Solving for the wavelength of light in the field-free region, we get

$$\lambda_f = \frac{T_f}{T_g}\lambda_g$$

Substituting the value of T_f from equation 3.64 into this we get

$$\lambda_f = \frac{T_g}{T_g}\left(1 + \frac{gy}{c^2}\right)\lambda_g$$

$$\lambda_f = \left(1 + \frac{gy}{c^2}\right)\lambda_g \tag{3.73}$$

Equation 3.73 gives the wavelength of light λ_f in the gravitational-field-free region. By the principle of equivalence, the wavelength of light emitted from an accelerated observer, accelerating with the constant acceleration a through a distance y is obtained from equation 3.73 as

$$\lambda_0 = \left(1 + \frac{ay}{c^2}\right)\lambda_a \tag{3.74}$$

where λ_0 is the wavelength of light that is observed in the region that is not accelerating, that is, the wavelength observed by an observer who is at rest. This result can be related to the velocity v that the accelerated observer attained during the constant acceleration by the kinematic equation

$$v^2 = v_0^2 + 2ay$$

Further assuming that the initial velocity v_0 is equal to zero and solving for the quantity ay we obtain

$$ay = \frac{v^2}{2}$$

Substituting this result into equation 3.74 we obtain

$$\lambda_0 = \left(1 + \frac{v_2}{2c^2}\right)\lambda_a \tag{3.75}$$

Using the binomial theorem in reverse as in equation 3.69,

$$\left(1 + \frac{v^2}{2c^2}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

equation 3.75 becomes

$$\lambda_0 = \frac{\lambda_a}{\sqrt{1 - v^2/c^2}}$$

Solving for $\lambda_{\mathbf{a}}$ we get

$$\lambda_a = \lambda_0\sqrt{1 - v^2/c^2} \tag{3.76}$$

But λ is a length, in particular $\lambda_{\mathbf{a}}$ is a length that is observed by the observer who has accelerated from 0 up to the velocity v and is usually referred to as L, whereas λ_0 is a length that is observed by an observer who is at rest relative to the measurement and is usually referred to as L_0. Hence, we can write equation 3.76 as

$$L = L_0\sqrt{1 - v^2/c^2} \tag{3.77}$$

But equation 3.77 is the Lorentz contraction of special relativity. *Hence, the Lorentz contraction is a special case of contraction of a length in a gravitational field, and by the principle of equivalence, a rod L_0 that is accelerated to the velocity v is contracted to the length L.* (That is, if a rod of length L_0 is at rest in a stationary spaceship, and the spaceship accelerates up to the velocity v, then the stationary observer on the earth would observe the contracted length L.) ***Hence, the acceleration of the rod is the basic physical principle underlying the length contraction***.

Thus, both the time dilation and length contraction of special relativity should be attributed to the warping of spacetime by the accelerating mass.

The warping of spacetime by the accelerating mass can be likened to the Doppler effect for sound. Recall from general physics that if a source of a sound wave is stationary, the sound wave propagates outward in concentric circles. When the sound source is moving, the waves are no longer circular but tend to bunch up in advance of the moving source. Since light does not require a medium for propagation, the Doppler effect for light is very much different. However, we can speculate that the warping of spacetime by the accelerating mass is comparable to the bunching up of sound waves in air. In fact, if we return to equation 3.63, for the gravitational red shift, and again, using the principle of equivalence, let $g = a$, and dropping the subscript f, this becomes

$$\nu = \nu_a \left(1 - \frac{ay}{c^2}\right) \tag{3.78}$$

Using the kinematic equation for constant acceleration, $ay = v^2/2$. Hence equation 3.78 becomes

$$\nu = \nu_a \left(1 - \frac{v^2}{2c^2}\right) \tag{3.79}$$

Again using the binomial theorem

$$\left(1 - \frac{v^2}{2c^2}\right) = \sqrt{1 - v^2/c^2}$$

Equation 3.78 becomes

$$\nu = \nu_a \sqrt{1 - v^2/c^2} \tag{3.80}$$

Equation 3.80 is called the transverse Doppler effect. It is a strictly relativistic result and has no counterpart in classical physics. The frequency $\nu_{\mathbf{a}}$ is the frequency of light emitted by a light source that is at rest in a coordinate system that is accelerating past a stationary observer, whereas ν is the frequency of light observed by the stationary observer. *Notice that the transverse Doppler effect comes directly from the gravitational red shift by using the equivalence principle.* Thus the transverse Doppler effect should be looked on as a frequency shift caused by accelerating a light source to the velocity v.

It is important to notice here that this entire derivation started with the gravitational red shift by the theory of the quanta, then the equivalence principle was used to obtain the results for an accelerating system. The Lorentz time dilation and length contraction came out of this derivation as a special case. Thus, the Lorentz equations should be thought of as kinematic equations, whereas the gravitational and acceleration results should be thought of as a dynamical result.

Time dilation and length contraction have always been thought of as only depending upon the velocity of the moving body and not upon its acceleration. As an example, in Wolfgang Rindler's book *Essential Relativity*,[1] he quotes results of experiments at the CERN laboratory where muons were accelerated. He states "that accelerations up to 10^{19} g (!) do not contribute to the muon time dilation." The only time dilation that could be found came from the Lorentz time dilation formula. They could not find the effect of the acceleration because they had it all the time. The Lorentz time dilation formula itself is a result of the acceleration. Remember, it is impossible to get a nonzero velocity without an acceleration.

In our study in chapter 2 we discussed how a very large collapsing star could become a black hole. Pursuing the equivalence principle further, if gravitational mass can warp spacetime into a black hole, can the singularity that would occur if a body could be accelerated to the velocity c, be considered as an accelerating black hole, and if so what implications would this have?

The Language of Physics

Photon
A small bundle of electromagnetic energy that acts as a particle of light. The photon has zero rest mass and its energy and momentum are determined in terms of the wavelength and frequency of the light wave (p.).

Photoelectric effect
Light falling on a metallic surface produces electrical charges. The photoelectric effect cannot be explained by classical electromagnetic theory. Einstein used the quantum theory to successfully explain this effect and won the Nobel Prize in physics. He said that a photon of light collides with an electron and imparts enough energy to it to remove it from its position in the metal (p.).

Principle of complementarity
The wave theory of light and the quantum theory of light complement each other. In a specific case, light exhibits either a wave nature or a particle nature, but never both at the same time (p.).

Compton effect
Compton bombarded electrons with photons and found that the scattered photon has a different wavelength than the incident light. The photon lost energy to the electron in the collision (p.).

de Broglie relation
de Broglie assumed that the same wave-particle duality associated with electromagnetic waves should also apply to particles. Thus, particles should also act

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1[1] Springer-Verlag, New York, 1979, Revised 2nd edition, p. 44.

as waves. The wave was first called a pilot wave, and then a matter wave. Today, it is simply called the wave function (p.).

Heisenberg uncertainty principle
The position and momentum of a particle cannot both be measured simultaneously with perfect accuracy. There is always a fundamental uncertainty associated with any measurement. This uncertainty is not associated with the measuring instrument. It is a consequence of the wave-particle duality of matter (p.).

Virtual particles
Ghostlike particles that exist around true particles. They exist by borrowing energy from the true particle, and converting this energy into mass. The energy must, however, be paid back before the time Δt, determined by the uncertainty principle, elapses. The virtual particles supply the force necessary to keep protons and neutrons together in the nucleus (p.).

Summary of Important Equations

Planck's relation $$E = nh\nu \quad (3.2)$$

Einstein's photoelectric equation $$KE_{max} = h\nu - W_0 \quad (3.10)$$

The work function $$W_0 = h\nu_0 \quad (3.11)$$

Properties of the photon

Rest mass $$m_0 = 0 \quad (3.14)$$

Energy $$E = h\nu \quad (3.6)$$

Relativistic mass $$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \quad (3.16)$$

Momentum $$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \quad (3.21)$$

Momentum of any particle $$p = \frac{\sqrt{E^2 - E_0^2}}{c} \quad (3.19)$$

Compton scattering formula $$(\lambda' - \lambda) = \frac{h}{m_0 c}(1 - \cos\phi) \quad (3.28)$$

de Broglie relation $$\lambda = \frac{h}{p} \quad (3.29)$$

The uncertainty principle

$$\Delta p \, \Delta x \geq \hbar \quad (3.44)$$

$$\Delta\theta \, \Delta L \geq \hbar \quad (3.51)$$

$$\Delta E\, \Delta t \geq \hbar \tag{3.55}$$

Angular momentum of a particle

$$L = rp \sin \theta \tag{3.47}$$
$$L = rp \tag{3.48}$$

Payback time for a virtual particle
$$\Delta t = \frac{\hbar}{(\Delta m)c^2} \tag{3.57}$$

Gravitational red shift

$$\nu_f = \nu_g \left(1 - \frac{gy}{c^2}\right) \tag{3.63}$$

$$T_f = T_g \left(1 + \frac{gy}{c^2}\right) \tag{3.64}$$

Slowing down of a clock in a gravitational field

$$\Delta t_f = \Delta t_g \left(1 + \frac{gy}{c^2}\right) \tag{3.65}$$

Slowing down of an accelerated clock

$$\Delta t_f = \Delta t_a \left(1 + \frac{ay}{c^2}\right) \tag{3.66}$$

$$\Delta t = \frac{\Delta t_a}{\sqrt{1 - v^2/c^2}} \tag{3.70}$$

Length contraction in a gravitational field
$$\lambda_f = \left(1 + \frac{gy}{c^2}\right)\lambda_g \tag{3.73}$$

Length contraction in an acceleration
$$\lambda_0 = \left(1 + \frac{ay}{c^2}\right)\lambda_a \tag{3.74}$$

$$L = L_0\sqrt{1 - v^2/c^2} \tag{3.77}$$

Questions for Chapter 3

*1. How would the world appear if Planck's constant h were very large? Describe some common occurrences and how they would be affected by the quantization of energy.

2. When light shines on a surface, is momentum transferred to the surface?

3. Could photons be used to power a spaceship through interplanetary space?

4. Should the concept of the cessation of all molecular motion at absolute zero be modified in view of the uncertainty principle?

5. Which photon, red, green, or blue, carries the most (a) energy and (b) momentum?

6. Discuss the entire wave-particle duality. That is, is light a wave or a particle, and is an electron a particle or a wave?

*7. Discuss the concept of determinism in terms of the uncertainty principle.

*8. Why isn't the photoelectric effect observed in all metals?

9. Ultraviolet light has a higher frequency than infrared light. What does this say about the energy of each type of light?

*10. Why can red light be used in a photographic dark room when developing pictures, but a blue or white light cannot?

Problems for Chapter 3

3.2 Blackbody Radiation

1. A weightless spring has a spring constant of 18.5 N/m. A 500-g mass is attached to the spring. It is then displaced 10.0 cm and released. Find (a) the total energy of the mass, (b) the frequency of the vibration, (c) the quantum number n associated with this energy, and (d) the energy change when the oscillator changes its quantum state by one value.

Diagram for problem 1.

2. Find the energy of a photon of light of 400.0-nm wavelength.

3. A radio station broadcasts at 92.4 MHz. What is the energy of a photon of this electromagnetic wave?

3.3 The Photoelectric Effect

4. The work function of a material is 4.52 eV. What is the threshold wavelength for photoelectronic emission?

5. The threshold wavelength for photoelectronic emission for a particular material is 518 nm. Find the work function for this material.

*6. Light of 546.0-nm wavelength is incident on a cesium surface that has a work function of 1.91 eV. Find (a) the frequency of the incident light, (b) the energy of the incident photon, (c) the maximum kinetic energy of the photoelectron, (d) the stopping potential, and (e) the threshold wavelength.

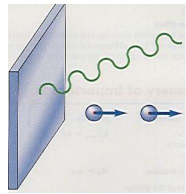

Diagram for problem 6.

3.4 The Properties of the Photon

7. A photon has an energy of 5.00 eV. What is its frequency and wavelength?
8. Find the mass of a photon of light of 500.0-nm wavelength.
9. Find the momentum of a photon of light of 500.0-nm wavelength.
10. Find the wavelength of a photon whose energy is 500 MeV.
11. What is the energy of a 650 nm photon?

3.5 The Compton Effect

12. An 80.0-KeV X ray is fired at a carbon target and Compton scattering occurs. Find the wavelength of the incident photon and the wavelength of the scattered photon for an angle of 40.0^0.

13. If an incident photon has a wavelength of 0.0140 nm, and is found to be scattered at an angle of 50.0^0 in Compton scattering, find the energy of the recoiling electron.

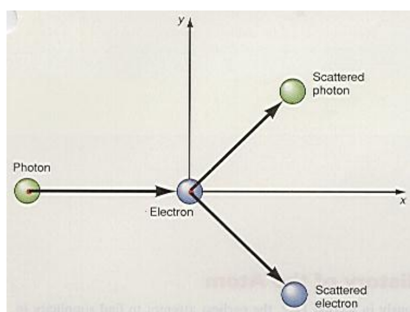

Diagram for problem 13.

14. In a Compton scattering experiment, 0.0400-nm photons are scattered by the target, yielding 0.0420-nm photons. What is the angle at which the 0.0420-nm photons are scattered?

3.6 The Wave Nature of Particles

15. Find the wavelength of a 4.60×10^{-2} kg golf ball moving at a speed of 60.0 m/s.
16. Find the wavelength of a proton moving at 10.0% of the speed of light.
17. Find the wavelength of an electron moving at 10.0% of the speed of light.
18. Find the wavelength of a 5.00-KeV electron.
19. Find the wavelength of an oxygen molecule at room temperature.
20. What is the frequency of the matter wave representing an electron moving at a speed of $2c/3$?

21. (a) Find the total energy of a proton moving at a speed of $c/2$. (b) Compute the wavelength of this proton.

3.8 The Heisenberg Uncertainty Principle

22. A 4.6×10^{-2} kg golf ball is in motion along the x-axis. If it is located at the position $x = 1.00$ m, with an uncertainty of 0.005 m, find the uncertainty in the determination of the momentum and velocity of the golf ball.

23. Find the minimum uncertainty in the determination of the momentum and speed of a 1300-kg car if the position of the car is to be known to a value of 10 nm.

24. The uncertainty in the position of a proton is 100 nm. Find the uncertainty in the kinetic energy of the proton.

3.9 Different Forms of the Uncertainty Principle

*25. The lifetime of an electron in an excited state of an atom is 10^{-8} s. From the uncertainty in the energy of the electron, determine the width of the spectral line centered about 550 nm.

Additional Problems

26. Approximately 5.00% of a 100-W incandescent lamp falls in the visible portion of the electromagnetic spectrum. How many photons of light are emitted from the bulb per second, assuming that the wavelength of the average photon is 550 nm?

Interactive Tutorials

27. *The photoelectric effect*. Light of wavelength $\lambda = 577.0$ nm is incident on a cesium surface. Photoelectrons are observed to flow when the applied voltage $V_0 = 0.250$ V. Find (a) the frequency ν of the incident photon, (b) the initial energy E of the incident photon, (c) the maximum kinetic energy KE_{max} of the photoelectrons, (d) the work function W_0 of cesium, (e) the threshold frequency ν_0, and (f) the corresponding threshold wavelength λ_0.

28. *The photoelectric effect*. Light of wavelength $\lambda = 460$ nm is incident on a cesium surface. The work function of cesium is $W_0 = 3.42 \times 10^{-19}$ J. Find (a) the frequency ν of the incident photon, (b) the initial energy E of the incident photon, (c) the maximum kinetic energy KE_{max} of the emitted photoelectrons, (d) the maximum speed v of the electron, (e) the threshold frequency ν_0, and (f) the corresponding longest wavelength λ_0 that will eject electrons from the metal.

29. *Properties of a photon*. A photon of light has a wavelength $\lambda = 420.0$ nm, find (a) the frequency ν of the photon, (b) the energy E of the photon, (c) the mass m of the photon, and (d) the momentum p of the photon.

30. *The Compton effect*. An x-ray photon of energy $E = 90.0$ KeV is fired at a carbon target and Compton scattering occurs at an angle $\phi = 30.0^0$. Find (a) the frequency ν of the incident photon, (b) the wavelength λ of the incident photon, and (c) the wavelength λ' of the scattered photon.

31. *Wave particle duality*. Using the concept of wave particle duality, calculate the wavelength λ of a golf ball whose mass $m = 4.60 \times 10^{-2}$ kg and is traveling at a speed $v = 60.0$ m/s.

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Chapter 4 Atomic Physics

"Nature resolves everything into it component atoms
and never reduces anything to nothing."
Lucretius (95 BC - 55 BC)

4.1 The History of the Atom

The earliest attempt to find simplicity in matter occurred in the fifth century B.C., when the Greek philosophers Leucippus and Democritus stated that matter is composed of very small particles called atoms. The Greek word for *atom* means "that which is indivisible." The concept of an atom of matter was to lie dormant for hundreds of years until 1803 when John Dalton, an English chemist, introduced his atomic theory of matter in which he proposed that to every known chemical element there corresponds an atom of matter. Today there are known to be 105 chemical elements. All other substances in the world are combinations of these elements.

The Greek philosophers' statement about atoms was based on speculation, whereas Dalton's theory was based on experimental evidence. Dalton's world of the atom was a simple and orderly place until some new experimental results appeared on the scene. M. Faraday performed experiments in electrolysis by passing electrical charges through a chemical solution of sodium chloride, NaCl. His laws of electrolysis showed that one unit of electricity was associated with one atom of a substance. He assumed that this charge was carried by the atom. A study of the conduction of electricity through rarefied gases led the English physicist J. J. Thomson, in 1898, to the verification of an independent existence of very small negatively charged particles. Even before this, in 1891, the Irish physicist George Stoney had made the hypothesis that there is a natural unit of electricity and he called it an electron. In 1896, Henri Becquerel discovered radiation from the atoms of a uranium salt.

The results of these experiments led to some inevitable questions. Where did these negatively charged particles, the electrons, come from? Where did the radiations from the uranium atom come from? The only place they could conceivably come from was from inside the atom. But if they came from within the atom, the atom could no longer be considered indivisible. That is, the "indivisible" atom must have some internal structure. Also, because the atom is observed to be electrically neutral, there must be some positive charge within it in order to neutralize the effect of the negative electrons. It had also been determined experimentally that these negative electrons were thousands of times lighter than the entire atom. Therefore, whatever positive charges existed within the atom, they must contain most of the mass of the atom.

J. J. Thomson proposed the first picture of the atom in 1898. He assumed that atoms are uniform spheres of positively charged matter in which the negatively charged electrons are embedded, figure 4.1. This model of the atom was called the plum pudding model because it resembled the raisins embedded in a plum pudding.

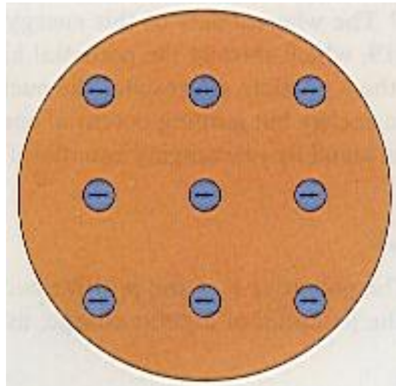

Figure 4.1 The plum pudding model of the atom.

That is, the electrons were like the raisins, and the pudding was the positively charged matter. But how are these electrons distributed within the atom? It became obvious that the only way to say exactly what is within the atom is to take a "look" inside the atom. But how do you "look" inside an atom?

In 1911, Hans Geiger (1882-1945) and Ernest Marsden, following a suggestion by Ernest Rutherford (1871-1937), bombarded atoms with alpha particles to "see" what was inside of the atom. The alpha particles, also written α particles, were high-energy particles emitted by radioactive substances that were discovered by Rutherford in 1899. These α particles were found to have a mass approximately four times the mass of a hydrogen atom and carried a positive charge of two units. (Today we know that the α particle is the nucleus of the helium atom.) Rutherford's idea was that the direction of motion of the α particle should be changed or deflected by the electrical charges within the atom. In this way, we can "see" within the atom. The experimental arrangement is illustrated in figure 4.2.

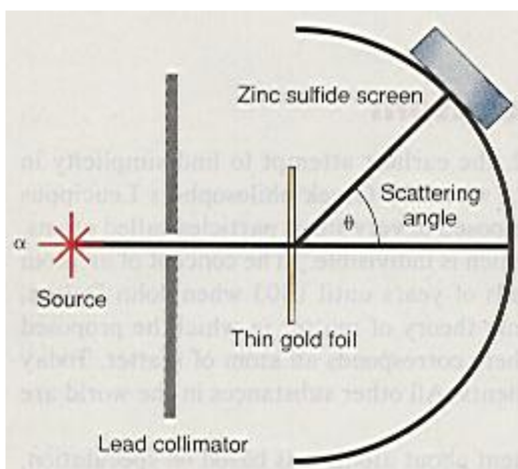

Figure 4.2 Rutherford scattering.

A polonium-214 source emits α particles of 7.68 MeV. A lead screen with a slit (the lead collimator) allows only those α particles that pass through the slit to fall on a very thin gold foil. It is expected that most of these α particles should go straight through the gold atoms and arrive at the zinc sulfide (ZnS) screen. When the α particle hits the ZnS screen a small flash of light is given off (called a *scintillation*). It is expected that some of these α particles should be slightly deflected by some of the positive charge of the atomic pudding. The ZnS screen can be rotated through any angle around the gold foil and can therefore observe any deflected α particles. *This process of deflection of the α particle is called Rutherford scattering.* The expected scattering from the distributed positive charge should be quite small and this was what at first was observed. Then Rutherford suggested that Geiger and Marsden should look for some scattering through large angles. To everyone's surprise, some α particles were found to come straight backward ($\theta \cong 180^0$). This back-scattering was such a shock to Rutherford that as he described it, "It was almost as incredible as if you fired a 15-inch shell at a piece of paper and it bounced off and came right back to you."

The only way to explain this back-scattering is to assert that the positive charge is not distributed over the entire atom but instead it must be concentrated in a very small volume. Thus, the experimental results of large-angle scattering are not consistent with the plum pudding model of an atom. Rutherford, therefore, proposed a new model of the atom, the *nuclear atom*. In this model, the atom consists of a very small, dense, positively charged nucleus. Because the negatively charged electrons would be attracted to the positive nucleus and would crash into it if they were at rest, it was necessary to assume that the electrons were in motion orbiting around the nucleus somewhat in the manner of the planets orbiting about the sun in the solar system. In 1919, Rutherford found this positive particle of the nucleus and named it the *proton*. The proton has a positive charge equal in magnitude to the charge on the electron (i.e., 1.60×10^{-19} C). The atom must contain the same number of protons as electrons in order to account for the fact that the atom is electrically neutral. The mass of the proton is 1.67×10^{-27} kg, which is 1836 times more massive than the electron.

A simple analysis allows us to determine the approximate dimensions of the nucleus. Consider a head-on collision between the α particle and the nucleus, as shown in figure 4.3. Because both particles are positive, there is an electrostatic

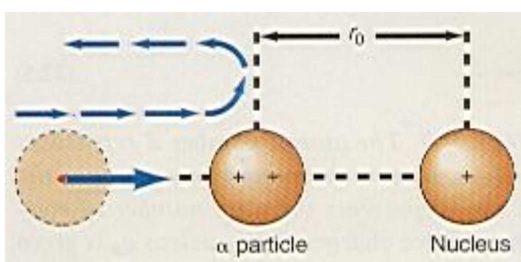

Figure 4.3 Approximate nuclear dimensions by a head-on collision.

force of repulsion between them. The α particle, as it approaches the nucleus, slows down because of the repulsion. Eventually it comes to a stop close to the nucleus, shown as the distance r_0 in the figure. The repulsive force now causes the α particle to accelerate away from the nucleus giving the back-scattering of 180^0. When the α particle left the source, it had a kinetic energy of 7.68 MeV. When it momentarily came to rest at the position r_0, its velocity was zero and hence its kinetic energy was also zero there. Where did all the energy go? The whereabouts of this energy can be determined by referring back to the potential hill of a positive point charge, such as a nucleus. As the α particle approaches the nucleus, it climbs up the potential hill, losing its kinetic energy but gaining potential energy. The potential energy that the α particle gains is found by

$$\text{PE} = q_{\text{alpha}}V$$

where q_{alpha} is the charge on the α particle. The potential V of the positive nucleus is given by the equation for the potential of a point charge, that is,

$$V = \frac{kq_{\mathbf{n}}}{r}$$

where $q_{\mathbf{n}}$ is the charge of the nucleus and k is the constant in Coulomb's law. Thus, the potential energy that the α particle gains as it climbs up the potential hill is

$$\text{PE} = \frac{kq_{\text{alpha}}q_{\text{n}}}{r}$$

When the α particle momentarily comes to rest at the distance r_0 from the nucleus, all its kinetic energy has been converted to potential energy. Equating the kinetic energy of the α particle to its potential energy in the field of the nucleus, we get

$$\text{KE} = \text{PE} = \frac{kq_{\text{alpha}}q_{\text{n}}}{r_0} \tag{4.1}$$

Because the kinetic energy of the α particle is known, equation 4.1 can be solved for $r_{0,}$ the approximate radius of the nucleus, to give

$$r_0 = \frac{kq_{\text{alpha}}q_{\text{n}}}{\text{KE}} \tag{4.2}$$

It had previously been determined that the charge on the α particle was twice the charge of the electron, that is,

$$q_{\text{alpha}} = 2e \tag{4.3}$$

To determine the charge of the nucleus, more detailed scattering experiments were performed with different foils, and it was found that the positive charge on the nucleus was approximately

$$q_n = \frac{Ae}{2} \tag{4.4}$$

where A is the mass number of the foil material. *The mass number A is the nearest whole number to the atomic mass of an element.* For example, the atomic mass of nitrogen (N) is 14.0067, its mass number A is 14. A new number, *the atomic number Z is defined from equation 4.4 as*[1]

$$Z = \frac{A}{2} \tag{4.5}$$

The atomic number of nitrogen is thus 14/2 = 7. *The atomic number Z represents the number of positive charges in the nucleus, that is, the number of protons in the nucleus. Because an atom is neutral, Z also represents the total number of electrons in the atom.* Using this notation the positive charge of the nucleus q_n is given by

$$q_n = Ze \tag{4.6}$$

Substituting equations 4.3 and 4.6 into equation 4.2 gives, for the value of r_0,

$$r_0 = \frac{k(2e)(Ze)}{\text{KE}}$$

or

$$r_0 = \frac{2kZe^2}{\text{KE}} \tag{4.7}$$

Example 4.1

The radius of the gold nucleus found by scattering. Find the maximum radius of a gold nucleus that is bombarded by 7.68-MeV α particles.

Solution

The maximum radius of the gold nucleus is found from equation 4.7 with the atomic number Z for gold equal to 79,

$$r_0 = \frac{2kZe^2}{\text{KE}}$$

$$r_0 = \left[\frac{2(9.00\times10^9 \text{ N m}^2/\text{C}^2)(79)(1.60\times10^{-19} \text{ C})^2}{7.68 \text{ MeV}}\right]\left(\frac{1 \text{ MeV}}{10^6 \text{ eV}}\right)\left(\frac{1 \text{ eV}}{1.60\times10^{-19} \text{ J}}\right)$$

$$= 2.96 \times 10^{-14} \text{ m}$$

1[1]Equation 4.5 is only correct for the lighter elements. For the heavier elements, there are more neutrons in the nucleus raising the value of the atomic mass and hence its mass number. We will discuss the neutron shortly.

Thus, the approximate radius of a gold nucleus is 2.96×10^{-14} m. The actual radius is somewhat less than this value. Although this calculation is only an approximation, it gives us the order of magnitude of the radius of the nucleus.

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It was already known from experimental data, that the radius of the atom $r_{\mathbf{a}}$ is of the order

$$r_{\mathbf{a}} = 10^{-10} \text{ m}$$

Comparing the relative size of the atom $r_{\mathbf{a}}$ to the size of the nucleus $r_{\mathbf{n}}$, we get

$$\frac{r_a}{r_n} = \frac{10^{-10} \text{ m}}{10^{-14} \text{ m}}$$
$$= 10^4$$

or

$$r_{\mathbf{a}} = 10^4\, r_{\mathbf{n}} = 10{,}000\, r_{\mathbf{n}} \tag{4.8}$$

That is, the radius of the atom is about 10,000 times greater than the radius of the nucleus.

More detailed scattering experiments have since led to the following approximate formula for the size of the nucleus

$$R = R_0 A^{1/3} \tag{4.9}$$

where A is again the mass number of the element and R_0 is a constant equal to 1.20×10^{-15} m.

Example 4.2

A more accurate value for the radius of a gold nucleus. Find the radius of the gold nucleus using equation 4.9.

Solution

The mass number A for gold is found by looking up the atomic mass of gold in the periodic table of the elements (appendix E). The atomic mass is 197.0. The mass number A for gold is the nearest whole number to the atomic mass, so $A = 197$. Now we find the radius of the gold nucleus from equation 4.9, as

$$R = R_0 A^{1/3}$$

$$= (1.20 \times 10^{-15}\ \text{m})(197)^{1/3}$$
$$= 0.698 \times 10^{-14}\ \text{m}$$

Notice that the radius obtained in this way is less than the value obtained by equation 4.7, which is, of course, to be expected.

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Because a nucleus contains Z protons, its mass should be Zm_p. Since the proton mass is so much greater than the electron mass, the mass of the atom should be very nearly equal to the mass of the nucleus. However, the atomic mass of an element was more than twice the mass of the Z protons. This led Rutherford in 1920 to predict the existence of another particle within the nucleus having about the same mass as the proton. Because this particle had no electric charge, Rutherford called it a neutron, for the neutral particle. In 1932, James Chadwick found this neutral particle, the neutron. The mass of the neutron was found to be $m_n = 1.6749 \times 10^{-27}$ kg, which is very close to the mass of the proton, $m_p = 1.6726 \times 10^{-27}$ kg.

With the finding of the neutron, the discrepancy in the mass of the nucleus was solved. *It can now be stated that an atom consists of Z electrons that orbit about a positive nucleus that contains Z protons and (A – Z) neutrons*.

Example 4.3

The number of electrons, protons, and neutrons in a gold atom. How many electrons, protons, and neutrons are there in a gold atom?

Solution

From the table of the elements, we see that the atomic number Z for gold is 79. Hence, there are 79 protons in the nucleus of the gold atom surrounded by 79 orbiting electrons. The mass number A for gold is found from the table of the elements to be 197. Hence, the number of neutrons in the nucleus of a gold atom is

$$A - Z = 197 - 79 = 118 \text{ neutrons}$$

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A very interesting characteristic of nuclear material is that all nuclei have the same density. For example, to find the density of nuclear matter, we use the definition of density,

$$\rho = \frac{m}{V}$$

Because the greatest portion of the matter of an atom resides in the nucleus, we can take for the mass of the nucleus

$$m = \frac{\text{Atomic mass}}{N_A} \tag{4.10}$$

where $N_A = 6.022 \times 10^{26}$ atoms/kmole $= 6.022 \times 10^{23}$ atoms/mole is Avogadro's number. Since the atomic mass, which has units of kg/kmole, is numerically very close to the mass number A, a dimensionless quantity, we can write the mass as

$$m = \frac{A(\text{kg/kmole})}{N_A} \tag{4.11}$$

We find the volume of the nucleus from the assumption that the atom is spherical, and hence

$$V = \frac{4}{3}\pi r^3 \tag{4.12}$$

Substituting the value for the radius of the nucleus found in equation 4.9 into equation 4.12, we get, for the volume of the nucleus,

$$V = \frac{4}{3}\pi(R_0 A^{1/3})^3$$

or

$$V = \frac{4}{3}\pi R_0^3 A \tag{4.13}$$

Substituting the mass from equation 4.11 and the volume from equation 4.13 back into the equation for the density, we get

$$\rho = \frac{m}{V} = \frac{A(\text{kg/kmole})/N_A}{(4/3)\pi R_0^3 A} \tag{4.14}$$

or

$$\rho = \frac{3(\text{kg/kmole})}{4\pi N_A R_0^3} \tag{4.15}$$

$$= \frac{3(\text{kg/kmole})}{4\pi(6.022 \times 10^{26}\ \text{atoms/kmole})(1.2 \times 10^{-15}\ \text{m})^3}$$

$$= 2.29 \times 10^{17}\ \text{kg/m}^3$$

Because the mass number A canceled out of equation 4.14, the density is the same for all nuclei. To get a "feel" for the magnitude of this nuclear density, note that a density of 2.29×10^{17} kg/m^3 is roughly equivalent to a density of a billion tons of matter per cubic inch, an enormously large number in terms of our usual experiences.

Now that the **Rutherford model of the atom** has been developed, let us look at some of its dynamical aspects. Let us consider the dynamics of the hydrogen atom. Because this new model of the atom is very similar to the planetary system of our solar system, we would expect the dynamics of the atom to be very similar to the dynamics of the planetary system. Recall that we assumed that a planet moved in a circular orbit and the necessary centripetal force for that circular motion was supplied by the gravitational force. In a similar analysis, let us now assume that the negative electron moves in a circular orbit about the positive nucleus. The Coulomb attractive force between the electron and the proton supplies the necessary centripetal force to keep the electron in its orbit. Therefore, equating the centripetal force F_c to the electric force F_e, we obtain

$$F_c = F_e \tag{4.16}$$

$$\frac{mv^2}{r} = \frac{k(e)(e)}{r^2} \tag{4.17}$$

Solving for the speed of the electron in its circular orbit, we get

$$v = \sqrt{\frac{ke^2}{mr}} \tag{4.18}$$

Thus, for a particular orbital radius r, there corresponds a particular velocity of the electron. This is, of course, the same kind of a relation found for the planetary case.

The total energy of the electron is equal to the sum of its kinetic and potential energy. Thus,

$$E = \text{KE} + \text{PE}$$

$$E = \tfrac{1}{2}mv^2 + \left(-\frac{ke^2}{r}\right) \tag{4.19}$$

where the negative potential energy of the electron follows from the definition of potential energy. (The zero of potential energy is taken at infinity, and since the electron can do work, as it approaches the positive nucleus, it loses some of its electric potential energy. Because it started with zero potential energy at infinity, its potential energy becomes more negative as it approaches the nucleus.) Substituting the speed of the electron for the circular orbit found in equation 4.18 into equation 4.19, we have, for the total energy,

$$E = \tfrac{1}{2}m\left(\frac{ke^2}{mr}\right) + \left(-\frac{ke^2}{r}\right)$$

$$E = \tfrac{1}{2}\frac{ke^2}{r} - \frac{ke^2}{r}$$

$$E = -\frac{ke^2}{2r} \tag{4.20}$$

The total energy of the electron is negative indicating that the electron is bound to the atom. Equation 4.20 says that the total energy of an electron in the Rutherford atom is not quantized — that is, the electron could be in any orbit of radius r and would have an energy consistent with that value of r.

From chemical analysis, it is known that it takes 13.6 eV of energy to ionize a hydrogen atom. This means that it takes 13.6 eV of energy to remove an electron from the hydrogen atom to infinity, where it would then have zero kinetic energy. Looking at this from the reverse process, it means that we need −13.6 eV of energy to bind the electron to the atom. This energy is called the *binding energy of the electron*. Knowing this binding energy permits us to calculate the orbital radius of the electron. Solving equation 4.20 for the orbital radius gives

$$r = -\frac{ke^2}{2E} \tag{4.21}$$

$$= -\left[\frac{(9.00\times10^{9}\ \text{N m}^2/\text{C}^2)(1.60\times10^{-19}\ \text{C})^2}{2(\text{-}13.6\ \text{eV})}\right]\left(\frac{1\ \text{eV}}{1.60\times10^{-19}\ \text{J}}\right)$$

$$= 5.29 \times 10^{-11}\ \text{m}$$

which is certainly the right order of magnitude.

Because the electron is in a circular orbit, it is undergoing accelerated motion. From the laws of classical electromagnetic theory, an accelerated electric charge should radiate electromagnetic waves. The frequency of these electromagnetic waves should correspond to the frequency of the accelerating electron. The frequency ν of the moving electron is related to its angular velocity ω by

$$\nu = \frac{\omega}{2\pi}$$

But the linear velocity v of the electron is related to its angular velocity by

$$v = \omega r$$

or

$$\omega = \frac{v}{r}$$

The frequency becomes

$$\nu = \frac{\omega}{2\pi} = \frac{v}{2\pi r} \tag{4.22}$$

Substituting for the speed v from equation 4.18, we get

$$\nu = \frac{v}{2\pi r} = \frac{1}{2\pi r}\sqrt{\frac{ke^2}{mr}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{ke^2}{mr^3}} \tag{4.23}$$

The frequency of the orbiting electron, and hence the frequency of the electromagnetic wave radiated, should be given by equation 4.23. Assuming the value of r computed above, the frequency becomes

$$\nu = \frac{1}{2\pi}\sqrt{\frac{ke^2}{mr^3}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{(9.00\times10^9 \text{ N m}^2/\text{C}^2)(1.60\times10^{-19}\text{ C})^2}{(9.11\times10^{-31}\text{ kg})(5.29\times10^{-10}\text{ m})^3}}$$

$$= 6.56 \times 10^{15} \text{ Hz}$$

This frequency corresponds to a wavelength of

$$\lambda = \frac{c}{\nu}$$

$$= \frac{3.00 \times 10^8 \text{ m/s}}{6.56 \times 10^{15} \text{ 1/s}}$$

$$= 4.57\times10^8 \text{ m}\left(\frac{1 \text{ nm}}{10^{-9}\text{ m}}\right)$$

$$= 45.7 \text{ nm}$$

The problem with this wavelength is that it is in the extreme ultraviolet portion of the electromagnetic spectrum, whereas some spectral lines of the hydrogen atom are known to be in the visible portion of the spectrum. An even greater discrepancy associated with Rutherford's model of the atom is that if the orbiting electron radiates electromagnetic waves, it must lose energy. If it loses energy by radiation, its orbital radius must decrease. For example, if the electron is initially in the state given by equation 4.20 as

$$E_i = \frac{ke^2}{2r_i} \tag{4.24}$$

After it radiates energy, it will have the smaller final energy E_f, given by

$$E_f = \frac{ke^2}{2r_f}$$

The energy lost by radiation is

$$E_f - E_i = \frac{ke^2}{2r_f} - \frac{ke^2}{2r_i}$$

$$= \frac{ke^2}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

But since the final state has less energy than the initial state

$$E_f - E_i < 0$$

this implies that the quantity

$$= \frac{ke^2}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right) < 0$$

The only way for this quantity to be less than zero is for

$$\left(\frac{1}{r_i} < \frac{1}{r_f}\right)$$

which requires that

$$r_f < r_i$$

That is, the final orbital radius is less than the initial radius. Hence, when the electron radiates energy, its orbital radius must decrease. But as the electron keeps orbiting, it keeps losing energy and its orbital radius keeps decreasing. As r decreases, the frequency of the electromagnetic waves increases according to equation 4.23 and its wavelength decreases continuously. Hence, the radiation from the atom should be continuous. Experimentally, however, it is found that the radiation from the atom is not continuous but is discrete. As the radius of the orbit keeps decreasing, the electron spirals into the nucleus and the atom should collapse. Because the entire world is made of atoms, it too should collapse. Since it does not, there is something very wrong with the dynamics of the Rutherford model of the atom.

4.2 The Bohr Theory of the Atom

The greatest difficulty with the Rutherford planetary model is that the accelerated electron should radiate a continuous spectrum of electromagnetic waves, thereby losing energy, and should thus, spiral into the nucleus. There is certainly merit in the planetary model, but it is not completely accurate. As we have seen, the search for truth in nature follows the path of successive approximations. Each approximation gets us closer to the truth, but we are still not there yet. How can this radiation problem of the atomic model be solved?

Niels Bohr (1885-1962), a young Danish physicist, who worked with J. J. Thomson, and then Rutherford, felt that the new success of the quantum theory by Planck and Einstein must be the direction to take in understanding the atom, that is, the atom must be quantized. But how?

In the **Bohr theory of the atom,** Bohr took the ingenious step of restricting the electron orbits to those for which the angular momentum is quantized. That is, *Bohr postulated that the electron could only be found in those orbits for which the angular momentum, L, is given by*

$$L = mvr = n\hbar \tag{4.25}$$

where n is called the principal quantum number and takes on the values 1, 2, 3, 4…. The value $\hbar = h/2\pi$ thus becomes a fundamental unit of angular momentum. *The consequence of this postulate is that the electron, which can now be looked on as a matter wave by the de Broglie hypothesis, can be represented in its orbit as a standing wave.* Consider the standing wave in figure 4.4(a). As you recall for a vibrating string, the nodes of a standing wave remain nodes for all time. The string cannot move up or down at that point, and, hence, cannot transmit any energy past that point. Thus, the standing wave does not move along the string, but is instead stationary or standing.

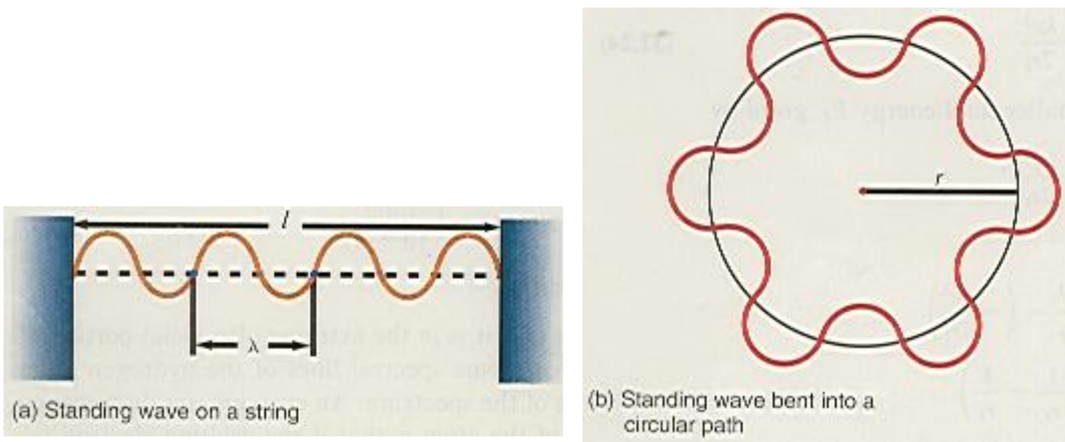

Figure 4.4 Standing wave of an electron in its orbit.

For the vibrating string fixed at both ends, the only waves that can stand for such a configuration are those for which the length of the string is equal to a multiple of a half wavelength, that is, for $l = n\lambda/2$. If the vibrating string is bent into a circle, figure 4.4(b), (perhaps this should be called a vibrating wire to justify bending it into a circle), the traveling waves are not reflected at a fixed boundary because there is now no fixed boundary. The waves keep passing around the circle. *The only waves that can stand in this circular configuration are those for which the length of the wire is a whole number of wavelengths.* Thus,

$$l = n\lambda \tag{4.26}$$

but the length of the wire is the circumference of the circle and is equal to $2\pi r$. Hence,

$$l = 2\pi r = n\lambda \tag{4.27}$$

The wavelength of the matter wave is given by the de Broglie relation as

$$\lambda = \frac{h}{p} = \frac{h}{mv} \tag{4.28}$$

Substituting equation 4.28 into equation 4.27, gives

$$2\pi r = \frac{nh}{mv}$$

or

$$mvr = \frac{nh}{2\pi} = n\hbar \tag{4.29}$$

But this is precisely the Bohr postulate for the allowed electron orbits, previously defined in equation 4.25. *Hence, Bohr's postulate of the quantization of the orbital angular momentum is equivalent to a standing matter wave on the electron orbit. But because standing waves do not change with time and thus, do not transmit energy, these matter waves representing the electron should not radiate electromagnetic waves. Thus, an electron in this prescribed orbit does not radiate energy and hence it does not spiral into the nucleus.* This state wherein an electron does not radiate energy is called a *stationary state*. With electrons in stationary states the atom is now stable.

The quantization of the orbital angular momentum displays itself as a quantization of the orbital radius, the orbital velocity, and the total energy of the electron. As an example, let us consider the dynamics of the Bohr model of the atom. Because it is basically still a planetary model, equations 4.16 and 4.17 still apply. However, equation 4.18 for the orbital speed is no longer applicable. Instead, we use equation 4.29 to obtain the speed of the electron as

$$v = \frac{n\hbar}{mr} \tag{4.30}$$

Substituting this value of v into equation 4.17, we get

$$F_c = F_e$$

$$\frac{mv^2}{r} = \frac{m}{r}\left(\frac{n\hbar}{mr}\right)^2 = \frac{ke^2}{r^2}$$

$$\frac{mn^2\hbar^2}{rm^2r^2} = \frac{ke^2}{r^2}$$

$$\frac{n^2\hbar^2}{rm} = ke^2$$

Solving for r, we get

$$r_n = \frac{\hbar^2}{kme^2}n^2 \tag{4.31}$$

Because of the n on the right-hand side of equation 4.31, the electron orbits are quantized. The subscript n has been placed on r to remind us that there is one value of r corresponding to each value of n. From the derivation, we see that r must be quantized in order to have standing or stationary waves.

Example 4.4

The radius of a Bohr orbit. Find the radius of the first Bohr orbit.

Solution

The radius of the first Bohr orbit is found from equation 4.31 with $n = 1$. Thus,

$$r_1 = \frac{\hbar^2}{kme^2}$$

$$= \frac{(1.0546\times10^{-34}\ \text{J s})^2}{(8.9878\times10^{9}\ \text{N m}^2/\text{C}^2)(9.1091\times10^{-31}\ \text{kg})(1.6022\times10^{-19}\ \text{C})^2}$$

$$= 5.219\times10^{-11}\ \text{m}\left(\frac{1\ \text{nm}}{10^{-9}\ \text{m}}\right)$$

$$= 0.0529\ \text{nm}$$

This is the radius of the electron orbit for the $n = 1$ state (called the *ground state*) and is called the *Bohr radius*.

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Now we can also write the radius of the nth orbit, equation 4.31, as

$$r_n = r_1 n^2 \tag{4.32}$$

Thus, the only allowed orbits are those for $r_n = r_1, 4r_1, 9r_1, 16r_1, \ldots$

The speed of the electron in its orbit can be rewritten by substituting equation 4.31 back into equation 4.30, yielding

$$v_n = \frac{n\hbar}{mr_n} = \frac{n\hbar}{m((\hbar)^2/kme^2)n^2}$$

or

$$v_n = \frac{ke^2}{n\hbar} \tag{4.33}$$

Because of the n on the right-hand side of equation 4.33, the speed of the electron is also quantized.

Example 4.5

The speed of an electron in a Bohr orbit. Find the speed of the electron in the first Bohr orbit.

Solution

The speed, found from equation 4.33 with $n = 1$, is

$$v_1 = \frac{ke^2}{\hbar}$$
$$= \frac{(9.00 \times 10^9 \text{ N m}^2/\text{C}^2)(1.60 \times 10^{-19} \text{ C})^2}{1.05 \times 10^{-34} \text{ J s}}$$
$$= 2.19 \times 10^6 \text{ m/s}$$

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The speed of the electron in higher orbits is obtained from equation 4.33 as

$$v_n = \frac{v_1}{n} \tag{4.34}$$

We will now see that the quantizing of the orbital radius and speed leads to the quantizing of the electron's energy. The total energy of the electron still follows from equation 4.19, but now $v = v_n$ and $r = r_n$. Thus,

$$E = \tfrac{1}{2}mv_n^2 + \left(-\frac{ke^2}{r_n}\right)$$

Substituting for r_n and v_n from equations 4.31 and 4.33, respectively, leads to

$$E = \frac{m}{2}\left(\frac{ke^2}{n\hbar}\right)^2 - \frac{ke^2}{\hbar^2 n^2 / kme^2}$$

$$= \frac{mk^2e^4}{2n^2\hbar^2} - \frac{mk^2e^4}{n^2\hbar^2}$$

$$E_n = -\frac{mk^2e^4}{2n^2\hbar^2} \tag{4.35}$$

Because of the appearance of the quantum number n in equation 4.35, the electron's energy is seen to be quantized. To emphasize this fact, we have placed the subscript n on E.

Example 4.6

The energy of an electron in a Bohr orbit. Find the energy of the electron in the first Bohr orbit.

Solution

The energy is found from equation 4.35 with $n = 1$ as follows

$$E_1 = -\frac{mk^2e^4}{2\hbar^2} \tag{4.36}$$

$$= -\frac{(9.1092\times10^{-31}\ \text{kg})(8.9878\times10^{9}\ \text{N m}^2/\text{C}^2)^2(1.6022\times10^{-19}\ \text{C})^4}{2(1.054\times10^{-34}\ \text{J s})^2}\left(\frac{1\ \text{eV}}{1.6022\times10^{-19}\ \text{J}}\right)$$

$$= -\ 13.6\ \text{eV}$$

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Thus, the energy of the electron in the first Bohr orbit is −13.6 eV. If this electron were to be removed from the atom, it would take 13.6 eV of energy. But the energy necessary to remove an electron from an atom is called the *ionization energy,* and it was previously known that the ionization energy of hydrogen was indeed 13.6 eV. Thus, the Bohr model of the atom seems to be on the right track in its attempt to represent the hydrogen atom.

When the electron is in the first Bohr orbit, it is said to be in the *ground state.* When it is in a higher orbit, it is said to be in an *excited state.* The energy of the electron in an excited state is given by equation 4.35, and in conjunction with equation 4.36, we can also write it as

$$E_n = -\frac{E_1}{n^2} \tag{4.37}$$

with E_1 = 13.6 eV. These different energy levels for the different states of the electron are drawn, in what is called an *energy-level diagram,* in figure 4.5. Note that as n gets larger the energy states get closer together until the difference between one energy state and another is so small that there are no longer any observable quantization effects. The energy spectrum is then considered continuous just as it is in classical physics. For positive values of energy ($E > 0$) the electron is no longer bound to the atom and is free to go anywhere.

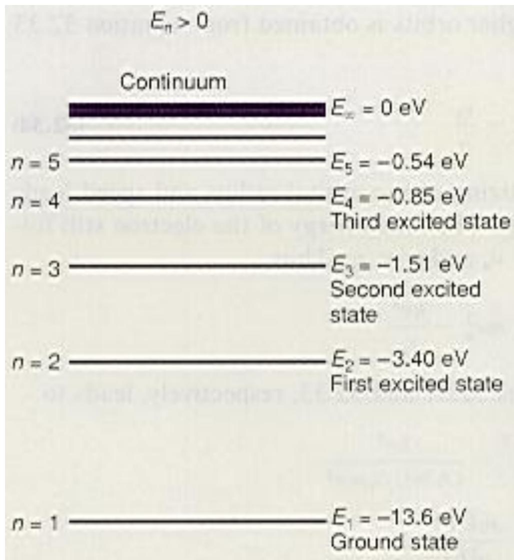

Figure 4.5 An energy-level diagram for the hydrogen atom.

4.3 The Bohr Theory and Atomic Spectra

Whenever sufficient energy is added to an electron it jumps to an excited state. The electron only stays in that excited state for a very short time (10^{-8} s). *Bohr next postulated that when the electron jumps from its initial higher energy state, E_i, to a final lower energy state, E_f, a photon of light is emitted in accordance with Einstein's relation*

$$h\nu = E_i - E_f \tag{4.38}$$

Using equation 4.37, we can write this as

$$h\nu = -\frac{E_1}{n_i^2} - \left(\frac{-E_1}{n_f^2}\right)$$

The frequency of the emitted photon is thus

$$\nu = \frac{E_1}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{4.39}$$

The wavelength of the emitted photon, found from $\nu = c/\lambda$, is

$$\nu = \frac{c}{\lambda} = \frac{E_1}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

or

$$\frac{1}{\lambda} = \frac{E_1}{hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{4.40}$$

Computing the value of E_1/hc gives

$$\begin{aligned}\frac{E_1}{hc} &= \left[\frac{13.6 \text{ eV}}{(6.6262\times10^{-34} \text{ J s})(2.9979\times10^{8} \text{ m/s})}\right]\left(\frac{1.6022\times10^{-19} \text{ J}}{1 \text{ eV}}\right)\\ &= (1.0969\times10^{7} \ \frac{1}{\text{m}})\left(\frac{10^{-9} \text{ m}}{1 \text{ nm}}\right)\\ &= 1.097 \times 10^{-2} \text{ (nm)}^{-1}\end{aligned}$$

We will come back to this shortly.

As seen in general physics, whenever white light is passed through a prism, it is broken up into a *continuous spectrum* of color from red through violet. On the other hand, if a gas such as hydrogen is placed in a tube under very low pressure and an electrical field is applied between two electrodes of the tube, the energy gained by the electrons from the field causes the electrons of the hydrogen atoms to jump to higher energy states. The gas glows with a characteristic color as the electrons fall back to the lower energy states. If the light from this spectral tube is passed through a prism or a diffraction grating, a line spectrum such as shown in figure 4.6 is found. That is, instead of the continuous spectrum of all the colors of

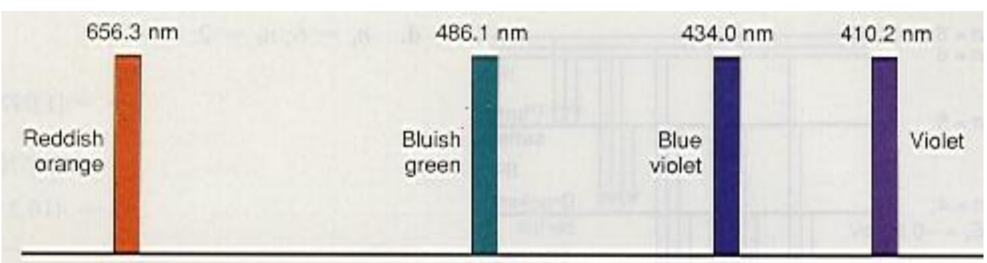

Figure 4.6 Line spectrum of hydrogen.

the rainbow, only a few discrete colors are found with the wavelengths indicated. The discrete spectra of hydrogen were known as far back as 1885 when Johann Jakob Balmer (1825-1898), a Swiss mathematician and physicist, devised the mathematical formula

$$\lambda = (364.56 \text{ nm})\frac{n^2}{n^2 - 4}$$

to describe the wavelength of the hydrogen spectrum. In 1896, the Swedish spectroscopist J. R. Rydberg (1853-1919) found the empirical formula

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \tag{4.41}$$

for n = 3, 4, 5, … , and where the constant R, now called the *Rydberg constant,* was given by $R = 1.097 \times 10^{-2}\ (\text{nm})^{-1}$. Comparing equation 4.41 with 4.40, we see that the Rydberg constant R is equal to the quantity E_1/hc and, if $n_f = 2$, the two equations are identical. Equation 4.41 was a purely empirical result from experiment with no indication as to why the spectral lines should be ordered in this way, whereas equation 4.40 is a direct result of the Bohr model of the hydrogen atom.

Example 4.7

Spectral lines with the Bohr model. Using the Bohr model of the hydrogen atom, determine the wavelength of the spectral lines associated with the transitions from the (a) $n_i = 3$ to $n_f = 2$ state, (b) $n_i = 4$ to $n_f = 2$ state, (c) $n_i = 5$ to $n_f = 2$ state, and (d) $n_i = 6$ to $n_f = 2$ state.

Solution

The wavelength of the spectral line is found from equation 4.40.

a. $n_i = 3$; $n_f = 2$:

$$\frac{1}{\lambda} = \frac{E_1}{hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$= [1.097 \times 10^{-2}\ (\text{nm})^{-1}]\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$$

$$= 1.5236 \times 10^{-3}\ (\text{nm})^{-1}$$

$$\lambda = 656.3\ \text{nm}$$

b. $n_i = 4$; $n_f = 2$:

$$\frac{1}{\lambda} = [1.097 \times 10^{-2}\ (\text{nm})^{-1}]\left(\frac{1}{2^2} - \frac{1}{4^2}\right)$$

$$= 2.0569 \times 10^{-3}\ (\text{nm})^{-1}$$

$$\lambda = 486.1\ \text{nm}$$

c. $n_i = 5$; $n_f = 2$:

$$\frac{1}{\lambda} = [1.097 \times 10^{-2}\ (\text{nm})^{-1}]\left(\frac{1}{2^2} - \frac{1}{5^2}\right)$$

$$= 2.3037 \times 10^{-3}\ (\text{nm})^{-1}$$

$$\lambda = 434.0\ \text{nm}$$

d. $n_i = 6$; $n_f = 2$:

$$\frac{1}{\lambda} = [1.097 \times 10^{-2}\ (\text{nm})^{-1}]\left(\frac{1}{2^2} - \frac{1}{6^2}\right)$$
$$= 2.4370 \times 10^{-3}\ (\text{nm})^{-1}$$
$$\lambda = 410.2\ \text{nm}$$

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Thus, the Bohr theory of the hydrogen atom agrees with the experimental values of the wavelengths of the spectral lines shown in figure 4.6. In fact, the Bohr formula is more complete than that given by the Rydberg formula. That is, the Rydberg series is associated with transitions to the $n = 2$ state. According to the Bohr formula, there should be spectral lines associated with transitions to the n_f = 1, 2, 3, 4, … , states. With such a prediction, it was not long before experimental physicists found these spectral lines. The reason they had not been observed before is because they were not in the visible portion of the spectrum. Lyman found the series associated with transitions to the ground state $n_f = 1$ in the ultraviolet portion of the spectrum. Paschen found the series associated with the transitions to the $n_f = 3$ state in the infrared portion of the spectrum. Brackett and Pfund found the series associated with the transitions to the $n_f = 4$ and $n_f = 5$ states, respectively, also in the infrared spectrum. The energy-level diagram and the associated spectral lines are shown in figure 4.7. Hence, the Bohr theory had great success in predicting the properties of the hydrogen atom.

Obviously, the Bohr theory of the atom was on the right track in explaining the nature and characteristics of the atom. However, as has been seen over and over again, physics arrives at a true picture of nature only through a series of successive approximations. The Bohr theory would be no different. As great as it was, it had its limitations. Why should the electron orbit be circular? The most general case would be elliptical. Arnold Sommerfeld (1868-1951) modified the Bohr theory to take into account elliptical orbits. The result increased the number of quantum numbers from one to two. With the advent of more refined spectroscopic equipment, it was found that some spectral lines actually consisted of two or more spectral lines. The Bohr theory could not account for this. Perhaps the greatest difficulty with the Bohr theory was its total inability to account for the spectrum of multielectron atoms. Also, some spectral lines were found to be more intense than others. Again, the Bohr theory could not explain why. Thus, the Bohr theory contains a great deal about the true nature of the atom, but it is not the complete picture. It is just a part of the successive approximations to a true picture of nature.

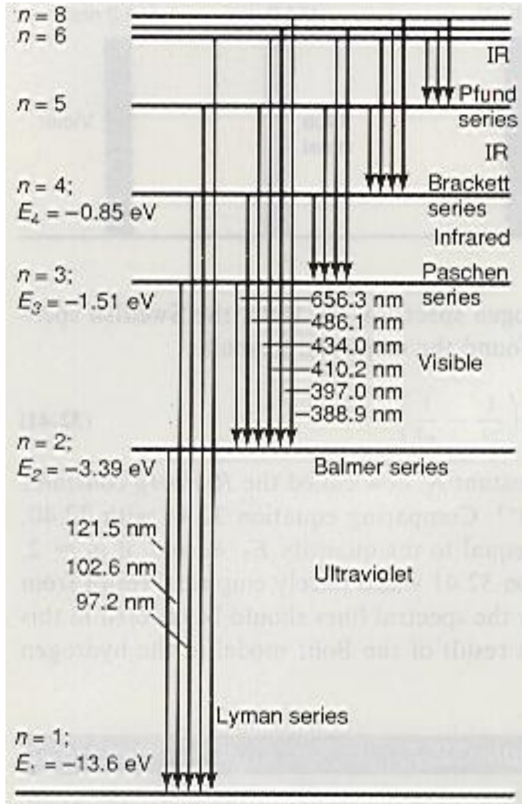

Figure 4.7 Associated spectrum for the energy-level diagram of hydrogen.

4.4 The Quantum Mechanical Model of the Hydrogen Atom

Obviously the Bohr theory of the hydrogen atom, although not wrong, was also not quite right. A new approach to the nature of the hydrogen atom was necessary. When de Broglie introduced his matter waves in 1924 to describe particles, it became necessary to develop a technique to find these matter waves mathematically. Erwin Schrödinger (1887-1961), an Austrian physicist, developed a new equation to describe these matter waves. This new equation is called the *Schrödinger wave equation.* (The Schrödinger wave equation is to quantum mechanics what Newton's second law is to classical mechanics. In fact, Newton's second law can be derived as a special case of the Schrödinger wave equation.) The solution of the wave equation is the wave function Ψ. For the **quantum mechanical model of the hydrogen atom,** the Schrödinger wave equation is applied to the hydrogen atom. It was found that it was necessary to have three quantum numbers to describe the electron in the hydrogen atom in this model,[2]

2[2] Actually the quantum mechanical model of the hydrogen atom requires four quantum numbers for its description. The fourth quantum number m_s, called the spin magnetic quantum number, is associated with the spin of the electron. The concept of the spin of the electron is introduced in section 4.7 and its effects are described there.

whereas the Bohr theory had required only one. The three quantum numbers are (1) the principal quantum number n, which is the same as that used in the Bohr theory; (2) the orbital quantum number l; and (3) the magnetic quantum number m_l. These quantum numbers are not completely independent; n can take on any value given by

$$n = 1, 2, 3, \ldots \tag{4.42}$$

Whereas l, the orbital quantum number, can only take on the values

$$l = 0, 1, 2, \ldots, (n - 1) \tag{4.43}$$

Thus, l is limited to values up to $n - 1$. The magnetic quantum number m_l can take on only the values given by

$$m_l = 0, \pm 1, \pm 2, \ldots, \pm l \tag{4.44}$$

Hence, m_l is limited to values up to $\pm l$. Let us now see a physical interpretation for each of these quantum numbers.

The Principal Quantum Number n

The principal quantum number n plays the same role in the quantum mechanical model of the hydrogen atom as it did in the Bohr theory in that it quantizes the possible energy of the electron in a particular orbit. The solution of the Schrödinger wave equation for the allowed energy values is

$$E_n = -\frac{k^2 e^4 m}{2\hbar^2}\frac{1}{n^2} \tag{4.45}$$

which we see has the same energy values as given by the Bohr theory in equation 4.35.

The Orbital Quantum Number l

In the Bohr theory, the angular momentum of the electron was quantized according to the relation $L = n\hbar$. The solution of the Schrödinger wave equation gives, for the angular momentum, the relation

$$L = \sqrt{l(l+1)}\,\hbar \tag{4.46}$$

where $l = 0, 1, 2, \ldots n - 1$.

Example 4.8

The angular momentum of an electron in a quantum mechanical model of the atom. Determine the angular momentum of an electron in the hydrogen atom for the orbital quantum numbers of (a) $l = 0$, (b) $l = 1$, (c) $l = 2$, and (d) $l = 3$.

Solution

The angular momentum of the electron is quantized according to equation 4.46 as

$$L = \sqrt{l(l+1)}\ \hbar$$

a. $l = 0$;

$$L = \sqrt{0(0+1)}\ \hbar$$
$$= 0$$

Thus for the $l = 0$ state, the angular momentum of the electron is zero. This is a very different case than anything found in classical physics. For an orbiting electron there must be some angular momentum, and yet for the $l = 0$ state, we get $L = 0$. Thus, the model of the atom with the electron orbiting the nucleus must now be considered questionable. We still speak of orbits, but they are apparently not the same simple concepts used in classical physics.

b. $l = 1$; $L = \sqrt{1(1+1)}\ \hbar = \sqrt{2}\ \hbar = 1.414\hbar$

c. $l = 2$; $L = \sqrt{2(2+1)}\ \hbar = \sqrt{6}\ \hbar = 2.449\hbar$

d. $l = 3$; $L = \sqrt{3(3+1)}\ \hbar = \sqrt{12}\ \hbar = 3.464\hbar$

Note that in the Bohr theory the angular momentum was a whole number times $\hbar$. Here in the quantum mechanical treatment the angular momentum is no longer a whole multiple of $\hbar$.

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The different angular momentum quantum states are usually designated in terms of the spectroscopic notation shown in table 4.1. The different states of the electron in the hydrogen atom are now described in terms of this spectroscopic notation in table 4.2. For the ground state, $n = 1$. However, because l can only take on values up to $n - 1$, l must be zero. Thus, the only state for $n = 1$ is the 1s or ground state of the electron. When $n = 2$, l can take on the values 0 and 1. Hence, there can be only a 2s and 2p state associated with $n = 2$. For $n = 3$, l can take on the values $l = 0$, 1, 2, and hence the electron can take on the states 3s, 3p, and 3d. In this way, for various values of n, the states in table 4.2 are obtained.

Table 4.1 Spectroscopic Notation for Angular Momentum			
Orbital Quantum Number l	Angular Momentum	State	Spectroscopic Name
0	0	s	Sharp
1	$\sqrt{2}\,\hbar$	p	Principal
2	$\sqrt{6}\,\hbar$	d	Diffuse
3	$2\sqrt{3}\,\hbar$	f	Fundamental
4	$\sqrt{20}\,\hbar$	g	
5	$\sqrt{30}\,\hbar$	h	

Table 4.2 Atomic States in the Hydrogen Atom						
	$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
$n = 1$	1s					
n = 2	2s	2p				
n = 3	3s	3p	3d			
n = 4	4s	4p	4d	4f		
n = 5	5s	5p	5d	5f	5g	
n = 6	6s	6p	6d	6f	6g	6h

The Magnetic Quantum Number m_l

Recall that angular momentum is a vector quantity and thus has a direction as well as a magnitude. We have just seen that the magnitude of the angular momentum is quantized. The result of the Schrödinger equation applied to the hydrogen atom shows that the direction of the angular momentum vector must also be quantized. The magnetic quantum number m_l specifies the direction of **L** by requiring the z-component of **L** to be quantized according to the relation

$$L_z = m_l \hbar \tag{4.47}$$

Quantization of the z-component of angular momentum specifies the direction of the angular momentum vector. As an example, let **L** be the angular momentum vector of the electron shown in figure 4.8. The z-component of **L** is found from the diagram to be

$$L_z = L \cos\theta \tag{4.48}$$

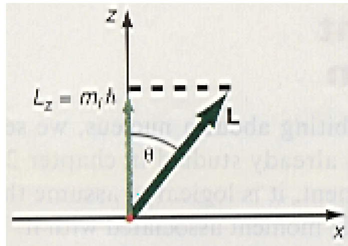

Figure 4.8 Direction of the angular momentum vector.

Substituting the value of L_z from equation 4.48 and the value of L from equation 4.46, gives

$$m_l \hbar = \sqrt{l(l+1)}\ \hbar \cos\theta$$

Solving for the angle θ that determines the direction of **L,** we get

$$\theta = \cos^{-1} \frac{m_l}{\sqrt{l(l+1)}} \tag{4.49}$$

Thus, for particular values of the quantum numbers l and m_l, the angle θ specifying the direction of **L** is determined.

Example 4.9

The direction of the angular momentum vector. For an electron in the state determined by $n = 4$ and $l = 3$, determine the magnitude and the direction of the possible angular momentum vectors.

Solution

For $n = 4$ and $l = 3$, the electron is in the 4*f* state. The magnitude of the angular momentum vector, found from equation 4.46, is

$$\begin{aligned} L &= \sqrt{l(l+1)}\ \hbar \\ &= \sqrt{3(3+1)}\ \hbar \\ &= 2\sqrt{3}\ \hbar \end{aligned}$$

The possible values of m_l, found from equation 4.44, are

$$m_l = 0,\ \pm 1,\ \pm 2,\ \pm 3$$

The angle θ, that the angular momentum vector makes with the *z*-axis, found from equation 4.49, is

$$\theta = \cos^{-1} \frac{m_l}{\sqrt{l(l+1)}}$$

$m_l = 0;$

$$\theta_0 = \cos^{-1} 0 = 90^0$$

$m_l = \pm 1;$

$$\theta_1 = \cos^{-1} \frac{+1}{2\sqrt{3}} = \pm 73.2^0$$

$m_l = \pm 2;$

$$\theta_2 = \cos^{-1} \frac{+2}{2\sqrt{3}} = \pm 54.7^0$$

$m_l = \pm 3;$

$$\theta_3 = \cos^{-1} \frac{+3}{2\sqrt{3}} = \pm 30.0^0$$

The various orientations of the angular momentum vector are shown in figure 4.9.

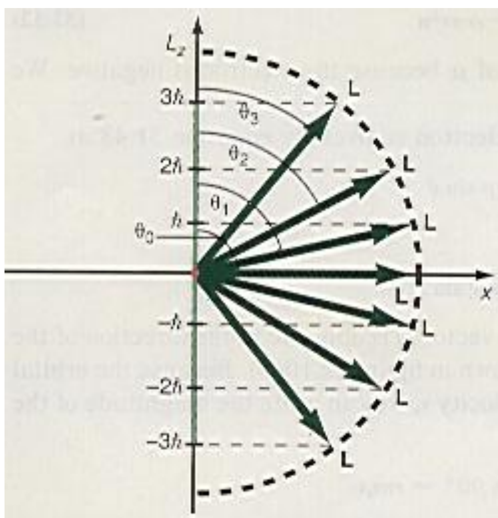

Figure 4.9 Quantization of the angular momentum vector.

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4.5 The Magnetic Moment of the Hydrogen Atom

Using the picture of an atom as an electron orbiting about a nucleus, we see that the orbiting electron looks like the current loop already studied in magnetism. Because a current loop has a magnetic dipole moment, it is logical to assume that the orbiting electron must also have a magnetic dipole moment associated with it.

Figure 4.10(a) shows a current loop, while figure 4.10(b) shows the electron in its orbit. The usual notation for a magnetic dipole moment in atomic physics is the

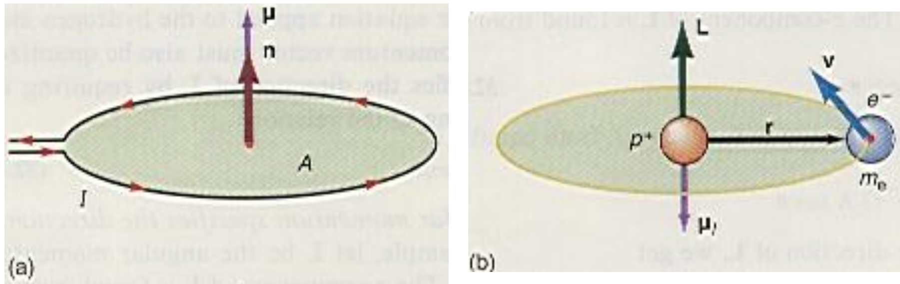

Figure 4.10 Orbital magnetic dipole moment.

Greek letter μ (mu). The magnetic dipole moment for the current loop in figure 4.10(a), becomes

$$\boldsymbol{\mu} = IA\mathbf{n}$$

Recall that I is the current in the loop, A is the area of the loop, and **n** is a unit vector that is normal to the current loop. The orbiting electron of figure 4.10(b) constitutes a current given by

$$I = \frac{-e}{T} \tag{4.50}$$

where $-e$ is the negative electronic charge and T is the time it takes for the electron to go once around its orbit. But the time to go once around its orbit is its period and, as seen previously, the period T is equal to the reciprocal of its frequency ν. That is, $T = 1/\nu$. Hence, equation 4.50 becomes

$$I = -e\nu \tag{4.51}$$

where ν is the number of times the electron circles its orbit in one second.

Thus, the orbiting electron looks like a current loop, with a current given by equation 4.51. We attribute to this current loop a magnetic dipole moment and call it the *orbital magnetic dipole moment*, designated by $\boldsymbol{\mu}_l$, and now given by

$$\boldsymbol{\mu}_l = IA\mathbf{n} = -e\nu A\mathbf{n}$$

Assuming the orbit to be circular, $A = \pi r^2$ and hence,

$$\boldsymbol{\mu}_l = -e\nu\pi r^2\mathbf{n} \tag{4.52}$$

Note that $\boldsymbol{\mu}_l$ is in the opposite direction of **n** because the electron is negative. We will return to this equation shortly.

The angular momentum of the electron is given by equation 3.47 as

$$L = rp \sin\theta$$

Since $p = m_e v$ we can write

$$L = r m_e v \sin\theta$$

where m_e is the mass of the electron. The vector **L** is opposite to the direction of the orbital magnetic dipole moment $\boldsymbol{\mu}_l$, as shown in figure 4.10(b). Because the orbital radius r is perpendicular to the orbital velocity **v**, we can write the magnitude of the angular momentum of the electron as

$$L = r m_e v \sin 90^0 = r m_e v$$

Using the same unit vector **n** to show the direction perpendicular to the orbit in figure 4.10(a), we can express the angular momentum of the orbiting electron as

$$\mathbf{L} = m_e r v \mathbf{n} \tag{4.53}$$

The speed of the electron in its orbit is just the distance s it travels along its arc divided by the time, that is,

$$v = \frac{s}{T} = \frac{2\pi r}{T} = 2\pi\nu r$$

Substituting this into equation 4.53, yields

$$\mathbf{L} = m_e r(2\pi\nu r)\mathbf{n} = 2\pi m_e \nu r^2 \mathbf{n} \tag{4.54}$$

Dividing equation 4.54 by $2m_e$, we get

$$\frac{\mathbf{L}}{2m_e} = \pi\nu r^2 \mathbf{n} \tag{4.55}$$

Returning to equation 4.52 and dividing by e, we get

$$-\frac{\boldsymbol{\mu}_l}{e} = \pi\nu r^2 \mathbf{n} \tag{4.56}$$

Comparing the right-hand sides of equations 4.55 and 4.56, we see that they are identical and can therefore be equated to each other giving

$$-\frac{\boldsymbol{\mu}_l}{e} = \frac{\mathbf{L}}{2m_e}$$

Solving for $\boldsymbol{\mu}_l$, we have

$$\boldsymbol{\mu}_l = -\frac{e}{2m_e}\mathbf{L} \tag{4.57}$$

Equation 4.57 is the orbital magnetic dipole moment of the electron in the hydrogen atom. That is, the orbiting electron has a magnetic dipole associated with it, and, as seen from equation 4.57, it is related to the angular momentum **L** of the electron. The quantity $e/2m_e$ is sometimes called the *gyromagnetic ratio*.

The magnitude of the orbital magnetic dipole moment is found from equation 4.57, with the value of L determined from equation 4.46. Hence,

$$\mu_l = \frac{eL}{2m_e} = \frac{e}{2m_e}\sqrt{l(l+1)}\ \hbar$$

$$\mu_l = \frac{e\hbar}{2m_e}\sqrt{l(l+1)} \qquad (4.58)$$

The quantity $e/2m_e$ is considered to be the smallest unit of magnetism, that is, an atomic magnet, and is called the *Bohr magneton*. Its value is

$$\frac{e\hbar}{2m_e} = \frac{(1.6021\times10^{-19}\ \text{C})(1.054\times10^{-34}\ \text{J s})}{2(9.1091\times10^{-31}\ \text{kg})}$$

$$= 9.274 \times 10^{-24}\ \text{A m}^2 = 9.274 \times 10^{-24}\ \text{J/T}$$

Example 4.10

The orbital magnetic dipole moment. Find the orbital magnetic dipole moment of an electron in the hydrogen atom when it is in (a) an s state, (b) a p state, and (c) a d state.

Solution

The magnitude of $\boldsymbol{\mu}_l$, found from equation 4.58, is

$$\mu_l = \frac{e\hbar}{2m_e}\sqrt{l(l+1)}$$

a. For an s state, $l = 0$,

$$\mu_l = 0$$

b. For a p state, $l = 1$,

$$\mu_l = \frac{e\hbar}{2m_e}\sqrt{1(1+1)}$$

$$= (9.274\times10^{-24}\ \text{J/T})\sqrt{2}$$

$$= 1.31 \times 10^{-23}\ \text{J/T}$$

c. For a d state, $l = 2$,

$$\mu_l = \frac{e\hbar}{2m_e}\sqrt{2(2+1)}$$

$$= (9.274 \times 10^{-24}\ \text{J/T})\sqrt{6}$$
$$= 2.26 \times 10^{-23}\ \text{J/T}$$

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The Potential Energy of a Magnetic Dipole in an External Magnetic Field
In general physics we saw that when a magnetic dipole μ is placed in an external magnetic field **B,** it experiences a torque given by

$$\tau = \mu B \sin\theta$$

This torque acts to rotate the dipole until it is aligned with the external magnetic field. Because the orbiting electron constitutes a magnetic dipole, if the hydrogen atom is placed in an external magnetic field, the orbital magnetic dipole of the atom rotates in the external field until it is aligned with it. Of course, since $\boldsymbol{\mu}_l$ is 180^0 opposite to **L,** the angular momentum of the atom, aligning the dipole in the field is equivalent to aligning the angular momentum vector of the atom. (Actually **L** is antiparallel to **B.**)

Because the natural position of $\boldsymbol{\mu}_l$ is parallel to the field, as shown in figure 4.11(a), work must be done to rotate $\boldsymbol{\mu}_l$ in the external magnetic field. When work was done in lifting a rock in a gravitational field, the rock then possessed potential

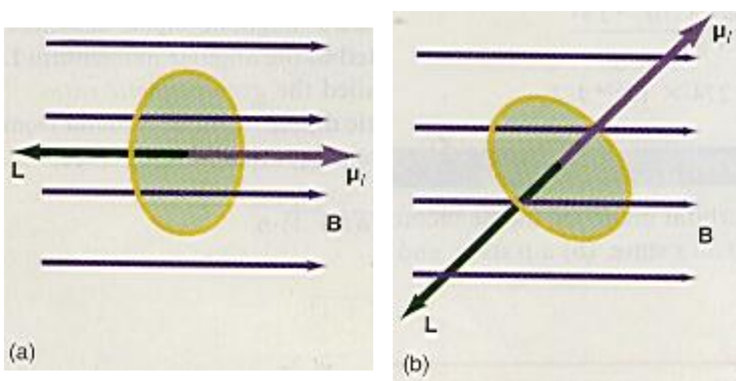

Figure 4.11 Orbital magnetic dipole in an external magnetic field **B**.

energy. In the same way, work done in rotating the dipole in the magnetic field shows up as potential energy of the dipole figure 4.11(b). That is, the electron now possesses an additional potential energy associated with the work done in rotating $\boldsymbol{\mu}_l$. It was shown in general physics, that the potential energy of a dipole in an external magnetic field **B** is given by

$$\text{PE} = -\mu_l B \cos\theta \tag{4.59}$$

Example 4.11

The potential energy of an orbital magnetic dipole moment. Find the potential energy of the orbital magnetic dipole in an external field when (a) it is antiparallel to **B** (i.e., $\theta = 180^0$), (b) it is perpendicular to **B** (i.e., $\theta = 90^0$), and (c) it is aligned with **B** (i.e., $\theta = 0$).

Solution

The potential energy of the dipole, found from equation 4.59, is

a.

$$PE = -\mu_l B \cos 180^0$$
$$PE = +\mu_l B$$

b.

$$PE = -\mu_l B \cos 90^0$$
$$PE = 0$$

c.

$$PE = -\mu_l B \cos 0^0$$
$$PE = -\mu_l B$$

Thus, the dipole has its highest potential energy when it is antiparallel (180^0), decreases to zero when it is perpendicular (90^0), and decreases to its lowest potential energy, a negative value, when it is aligned with the magnetic field, $\theta = 0^0$. This is shown in figure 4.12. So, just as the rock falls from a position of high potential energy to the ground where it has its lowest potential energy, the dipole, if given a slight push to get it started, rotates from its highest potential energy (antiparallel) to its lowest potential energy (parallel).

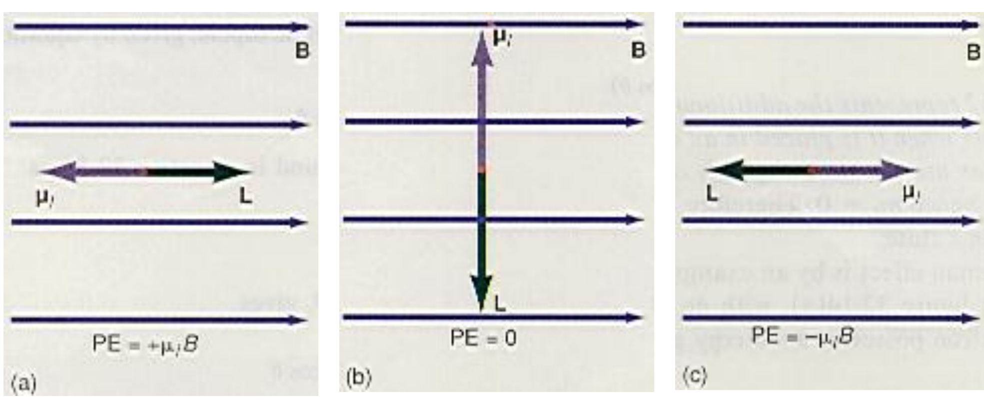

Figure 4.12 Potential energy of a dipole in an external magnetic field.

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4.6 The Zeeman Effect

The fact that there is a potential energy associated with a magnetic dipole placed in an external magnetic field has an important consequence on the energy of a particular atomic state, because the energy of a particular quantum state can change because of the acquired potential energy of the dipole. This acquired potential energy manifests itself as a splitting of a single energy state into multiple energy states, with a consequent splitting of the spectral lines associated with the transitions from these multiple energy states to lower energy states. The entire process is called the **Zeeman effect** after the Dutch physicist Pieter Zeeman (1865-1943) who first observed the splitting of spectral lines into several components when the atom was placed in an external magnetic field. Let us now analyze the phenomenon.

Let us begin by orienting an ordinary magnetic dipole, as shown in figure 4.13(a). A uniform magnetic field **B** is then turned on, as shown in figure 4.13(b). A torque acts on the dipole and the dipole becomes aligned with the field as expected. If the orbital magnetic dipole is oriented in the same way, figure 4.13(c),

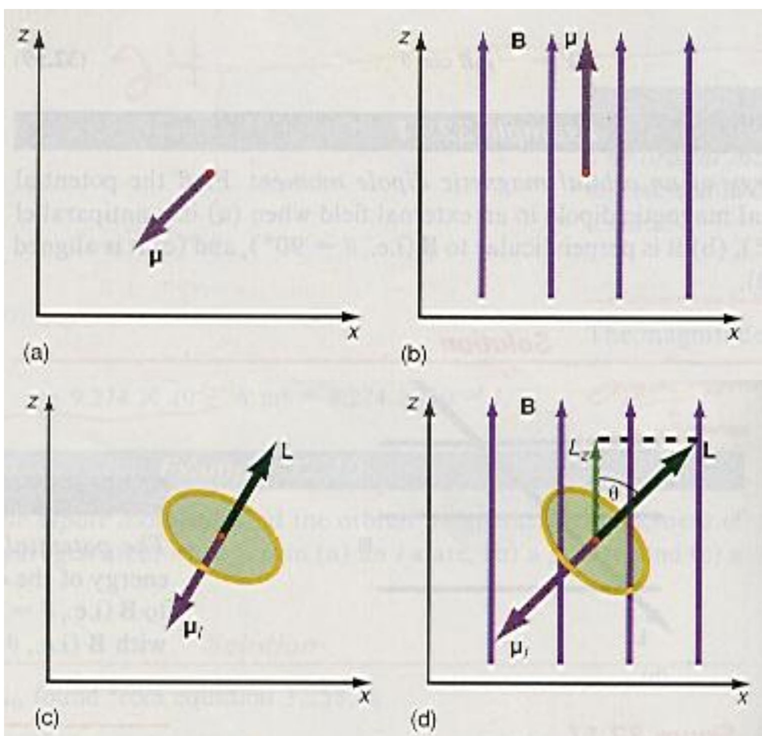

Figure 4.13 Orientation of the magnetic dipole in an external magnetic field.

and then the magnetic field **B** is turned on, a torque acts to align $\boldsymbol{\mu}_l$. But the quantum conditions say that the angular momentum vector **L** can only be oriented such that its z-component L_z must be equal to $m_l\hbar$, equation 4.47. Hence, the dipole cannot rotate completely to align itself with **B,** but stops rotating at a position, such as in figure 4.13(d), where $L_z = m_l\hbar$. The orbital magnetic dipole has the potential energy given by equation 4.59 when stopped in this position.

This is strictly a quantum mechanical phenomenon, not found in classical physics. Its analogue in classical physics would be dropping a rock in the gravitational field, where the rock falls a certain distance and then comes to a stop some distance above the surface of the earth. This is an effect never observed classically.

The potential energy of the orbital magnetic dipole, given by equation 4.59, is

$$\text{PE} = -\,\mu_l B \cos\theta$$

But the orbital magnetic dipole moment was found in equation 4.57 as

$$\boldsymbol{\mu}_l = -\frac{e}{2m_e}\,\mathbf{L}$$

Substituting equation 4.57 into equation 4.59, gives

$$\text{PE} = +\frac{e}{2m_e}\,LB\cos\theta \tag{4.60}$$

But,

$$LB\cos\theta = B(L\cos\theta) \tag{4.61}$$

And, as seen from figure 4.13(d),

$$L\cos\theta = L_z$$

Substituting this into equation 4.60, gives

$$\text{PE} = \frac{e}{2m_e}\,L_zB$$

Finally, substituting for $L_z = m_l\hbar$ we get

$$\text{PE} = m_l\frac{e\hbar}{2m_e}\text{B} \tag{4.62}$$

where $m_l = 0, \pm1, \pm2, \ldots, \pm l$. *Equation 4.62 represents the additional energy that an electron in the hydrogen atom can possess when it is placed in an external magnetic field. Hence, the energy of a particular atomic state depends on m_l as well as n.* Also, note that for s states, $l = 0$, and hence $m_l = 0$. Therefore, there is no potential energy for the dipole when it is in an s state.

Perhaps the best way to explain the Zeeman effect is by an example. Suppose an electron is in the $2p$ state, as shown in figure 4.14(a), with no applied external

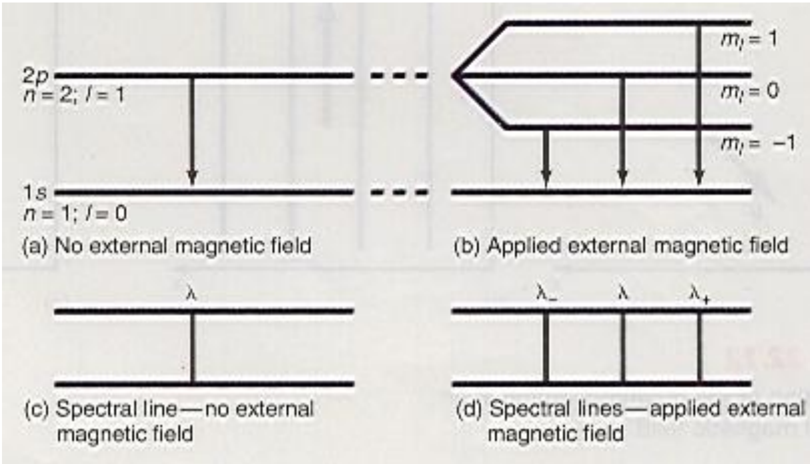

Figure 4.14 The Zeeman effect.

magnetic field. In the 2*p* state, the electron possesses an energy given by equation 4.45 as

$$E_2 = -\frac{k^2e^4m}{2\hbar^2}\frac{1}{2^2} = \frac{E_1}{4} \tag{4.63}$$

When the electron drops to the 1*s* state it has the energy

$$E_1 = -\frac{k^2e^4m}{2\hbar^2} = -13.6 \text{ eV}$$

and has emitted a photon of energy

$$h\nu = E_2 - E_1 = \frac{E_1}{4} - E_1 = -\frac{3E_1}{4}$$

with a frequency of

$$\nu_0 = -\frac{3E_1}{4h} \tag{4.64}$$

and a wavelength of

$$\lambda = \frac{c}{\nu_0} = \frac{ch}{\frac{3}{4}E_1} \tag{4.65}$$

Example 4.12

An electron drops from the 2p state to the 1s state. Find (a) the energy of an electron in the 2*p* state, (b) the energy lost by the electron as it drops from the 2*p* state to the 1*s* state, (c) the frequency of the emitted photon, and (d) the wavelength of the emitted photon.

Solution

a. The energy of the electron in the $2p$ state, given by equation 4.63, is

$$E_2 = \frac{E_1}{4} = -\frac{13.6 \text{ eV}}{4} = -3.40 \text{ eV}$$

b. The energy lost by the electron when it drops from the $2p$ state to the $1s$ state is found from

$$\Delta E = E_2 - E_1 = -3.40 \text{ eV} - (-13.6 \text{ eV}) = 10.2 \text{ eV}$$

c. The frequency of the emitted photon, found from equation 4.64, is

$$\nu_0 = -\frac{3E_1}{4h}$$
$$= \left[\frac{-3(-13.6 \text{ eV})}{4(6.63\times10^{-34} \text{ J s})}\right]\left(\frac{1.60\times10^{-19} \text{ J}}{1 \text{ eV}}\right)$$
$$= 2.46 \times 10^{15} \text{ Hz}$$

d. The wavelength of the emitted photon, found from equation 4.65, is

$$\lambda = \frac{c}{\nu_0} = \frac{ch}{\frac{3}{4}E_1}$$
$$\lambda_0 = \frac{c}{\nu_0} = \frac{3.00\times10^{8} \text{ m/s}}{2.46\times10^{15} \text{ 1/s}} = 1.22\times10^{-7} \text{ m}$$

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When the magnetic field is turned on, the electron in the $2p$ state acquires the potential energy of the dipole and now has the energy

$$E_2' = E_2 + \text{PE} = E_2 + m_l\frac{(e\hbar)B}{2m_e} \qquad (4.66)$$

But for a p state, $l = 1$ and m_l then becomes equal to 1, 0, and −1. Thus, there are now three energy levels associated with E_2' because there are now three values of m_l. Therefore,

$$E_2' = E_2 + \frac{(e\hbar)B}{2m_e} \qquad (4.67)$$

$$E_2' = E_2 \tag{4.68}$$

$$E_2' = E_2 - \frac{(e\hbar)B}{2m_e} \tag{4.69}$$

These energy states are shown in figure 4.14(b). Thus, the application of the magnetic field has split the single 2*p* state into 3 states. Since there are now three energy states, an electron can be in any one of them and hence, there are now three possible transitions to the ground state, where before there was only one. Corresponding to each of these three transitions are three spectral lines, as shown in figure 4.14(d). *Thus, the application of the magnetic field splits the single spectral line of figure 4.14(c) into the three spectral lines of figure 4.14(d).* The emitted photon associated with the transition from the $m_l = +1$ state is

$$h\nu_+ = E_{2+}' - E_1 = E_2 + \frac{e\hbar B}{2m_e} - E_1$$

$$= \frac{E_1}{4} - E_1 + \frac{e\hbar B}{2m_e} = -\frac{3E_1}{4} + \frac{e\hbar B}{2m_e}$$

Using equation 4.64 the frequency of this spectral line becomes

$$\nu_+ = \nu_0 + \frac{e\hbar B}{2m_e h}$$

The wavelength of the spectral line is given by

$$\lambda_+ = \frac{c}{\nu_+} = \frac{c}{\nu_0 + e\hbar B/2m_e h} \tag{4.70}$$

Comparing equation 4.70 with equation 4.65, we see that λ_+ is slightly smaller than the original wavelength λ_0.

The transition from the $m_l = 0$ state is the same as the original transition from the 2*p* state because the electron has no potential energy associated with the magnetic dipole for $m_l = 0$. Thus, the spectral line is of the same wavelength λ_0 observed in the nonsplit spectral line.

The transition from the $m_l = -1$ state to the 1*s* state emits a photon of energy,

$$h\nu_- = E_{2-}' - E_1 = E_2 - \frac{e\hbar B}{2m_e} - E_1$$

$$= \frac{E_1}{4} - E_1 - \frac{e\hbar B}{2m_e} = -\frac{3E_1}{4} - \frac{e\hbar B}{2m_e}$$

Using equation 4.64, the frequency of this spectral line becomes

$$\nu_- = \nu_0 - \frac{e\hbar B}{2m_e h}$$

The wavelength of the spectral line is

$$\lambda_- = \frac{c}{\nu_-} = \frac{c}{\nu_0 - e\hbar B/2m_e h} \tag{4.71}$$

Comparing equation 4.71 with equation 4.65, we see that the wavelength λ_- is slightly larger than the original wavelength λ_0.

It turns out that all transitions from split states are not necessarily observed. Certain transitions are forbidden, and the allowed transitions are given by a set of selection rules on the allowed values of the quantum numbers. *Allowed transitions are possible only for changes in states where*

Selection Rules
$$\begin{aligned} \Delta l &= \pm 1 \\ \Delta m_l &= 0, \ \ \pm 1 \end{aligned} \tag{4.72}$$

Note that these selection rules were obeyed in the preceding example.

The selection rule requiring l to change by ± 1 means that the emitted photon must carry away angular momentum equal to the difference between the angular momentum of the atom's initial and final states.

4.7 Electron Spin

The final correction to the model of the atom assumes that the electron is not quite a point charge, but its charge is distributed over a sphere. As early as 1921 A. H. Compton suggested that the electron might be a spinning particle. The Dutch-American physicists, Samuel Goudsmit and George Uhlenbeck inferred, in 1925, that the electron did spin about its own axis and because of this spin had an additional angular momentum, **S,** associated with this spin. Thus, this semiclassical model of the atom has an electron orbiting the nucleus, just as the earth orbits the sun, and the electron spinning on its own axis, just as the earth does about its axis. The model of the electron as a rotating charged sphere gives rise to an equivalent current loop and hence a magnetic dipole moment $\boldsymbol{\mu}_s$ associated with this spinning electron.

Associated with the orbital angular momentum **L** was the orbital quantum number l. Similarly, associated with the spin angular momentum **S** is the spin quantum number s. Whereas l could take on the values l = 0, 1, 2, …, n − 1, s can only take on the value

$$s = \frac{1}{2} \tag{4.73}$$

Similar to the magnitude of the orbital angular momentum given in equation 4.46, the magnitude of the spin angular momentum is given by

$$S = \sqrt{s(s+1)}\ \hbar \tag{4.74}$$

Because s can only take on the value 1/2, the magnitude of the spin angular momentum S can only be

$$S = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)}\ \hbar = \tfrac{1}{2}\sqrt{3}\ \hbar \tag{4.75}$$

Just as the direction of the orbital angular momentum vector was quantized according to equation 4.47, the z-component of the spin angular momentum is quantized to

$$S_z = m_s \hbar \tag{4.76}$$

where m_s is called the spin magnetic quantum number. Just as the angular momentum vector **L** could have $2l + 1$ directions, that is, $m_l = 0, \pm 1, \pm l$, the spin angular momentum vector **S** can have $2s + 1$ directions, that is, $2s + 1 = 2(1/2) + 1 = 2$ directions specified by $m_s = ½$ and $m_s = -½$. Thus, the z-component of the spin angular momentum can only be

$$S_z = \pm \frac{\hbar}{2} \tag{4.77}$$

The only two possible spin angular momentum orientations are shown in figure 4.15. When $m_s = +1/2$, the electron is usually designated as spin-up, while $m_s = -½$ is referred to as spin-down. The state of any electron in an atom is now specified by the four quantum numbers n, l, m_l, and m_s.

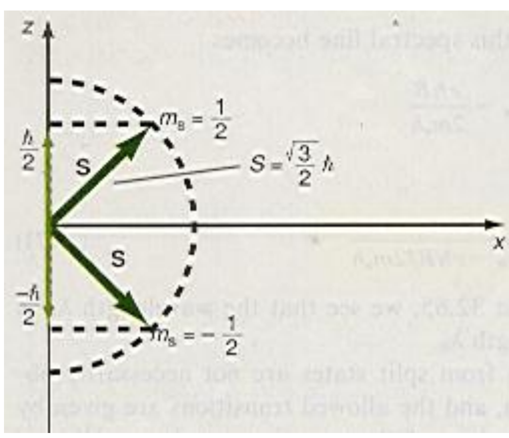

Figure 4.15 Orientations of the spin angular momentum vector.

We should note that the semiclassical picture of the spinning electron is not quite correct. Using the picture of a spinning sphere, we can find its angular momentum about its own axis from the study of rotational motion as

$$L = I\omega = \left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)$$
$$= \frac{2}{5}mrv$$

where r is the radius of the electron, which is of the order of 10^{-15} m. Because the spin angular momentum has only the one value given by equation 4.75, for these angular momenta to be equal, **L = S,** or

$$\tfrac{2}{5}mrv = \tfrac{1}{2}\sqrt{3}\,\hbar$$

The speed of a point on the surface of the spinning electron would have to be

$$v = \frac{5\sqrt{3}\,\hbar}{4mr}$$
$$= \frac{5\sqrt{3}\,(1.0546\times10^{-34}\text{ J s})}{4(9.11\times10^{-31}\text{ kg})(1.00\times10^{-15}\text{ m})}$$
$$= 2.5 \times 10^{11}\text{ m/s}$$

But this is a velocity greater than the velocity of light, which cannot be. Hence, the classical picture of the charged rotating sphere cannot be correct. However, in 1928, Paul A. M. Dirac (1902-1984) joined together the special theory of relativity and quantum mechanics, and from this merger of the two theories found that the electron must indeed have an intrinsic angular momentum that is the same as that given by the semiclassical spin angular momentum. This angular momentum is purely a quantum mechanical effect and although of the same magnitude as the spin angular momentum, it has no classical analogue. However, because the value of the angular momentum is the same, it is still customary to speak of the spin of the electron.

Just as the orbital angular momentum has an orbital magnetic dipole moment $\boldsymbol{\mu}_l$ given by equation 4.57, the spin angular momentum has a spin magnetic dipole moment given by

$$\mu_s = -2.0024\left(\frac{e}{2m_e}\right)S \tag{4.78}$$

and the spin magnetic dipole moment is shown in figure 4.16. In what follows, we will round off the value 2.0024 in equation 4.78 to the value 2.

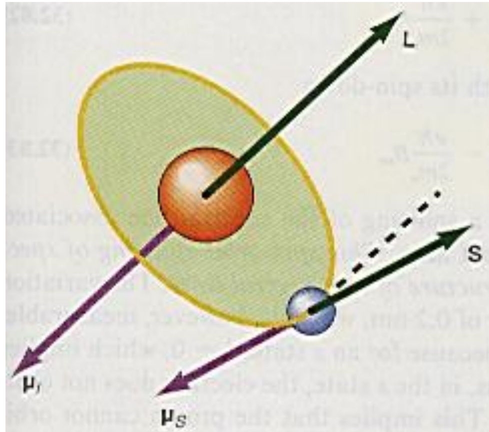

Figure 4.16 Orbital and spin angular momentum vectors and their associated dipole moments.

When the orbital magnetic dipole $\boldsymbol{\mu}_l$ was placed in a magnetic field, a torque acted on $\boldsymbol{\mu}_l$ trying to align it with the magnetic field. The space quantization of **L** made it impossible for **L** to become aligned, and hence, the electron had the potential energy given by equation 4.62. In the same way, the spin magnetic dipole moment $\boldsymbol{\mu}_s$ should try to align itself in any magnetic field and because of the space quantization of the spin angular momentum, the electron should have the potential energy

$$\text{PE} = -\mu_s B \cos\theta \tag{4.79}$$

Substituting equation 4.78 into equation 4.79, gives

$$\text{PE} = +2\left(\frac{e}{2m_e}\right) SB\cos\theta \tag{4.80}$$

If there is an applied magnetic field **B,** the electron acquires the additional potential energy given by equation 4.80. However, if there is no applied magnetic field, this potential energy term is still present, because from the frame of reference of the electron, the proton is in orbit about the electron, figures 4.17(a) and 4.17(b). The revolving proton constitutes a current loop and produces a magnetic field $\mathbf{B}_{so}$ at the location of the electron. This magnetic field interacts with the spin magnetic dipole moment $\boldsymbol{\mu}_s$ as given by equation 4.80. *The interaction of the spin magnetic dipole with the magnetic field* $\boldsymbol{B}_{so}$ *produced by the orbiting proton is called the spin-orbit interaction.*

We can now write equation 4.80 as

$$\text{PE} = +2\left(\frac{e}{2m_e}\right) SB_{so}\cos\theta$$

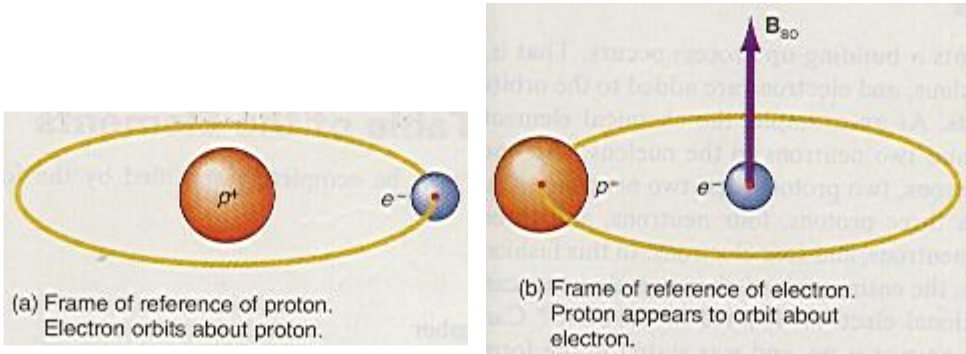

Figure 4.17 The spin-orbit interaction.

But $S\cos\theta = S_z$, the z-component of the spin angular momentum vector. Hence,

$$\text{PE} = +2\left(\frac{e}{2m_e}\right)S_z B_{so}$$

But $S_z = \pm\hbar/2$ from equation 4.77, thus

$$\text{PE} = +2\left(\frac{e}{2m_e}\right)\frac{\pm\hbar}{2}B_{so}$$

or the acquired potential energy of an electron caused by the spin-orbit interaction is

$$\text{PE} = \pm\left(\frac{e\hbar}{2m_e}\right)B_{so} \tag{4.81}$$

Hence, every quantum state, except *s* states, splits into two energy states, one corresponding to the electron with its spin-up

$$E_1 = E_0 + \left(\frac{e\hbar}{2m_e}\right)B_{so} \tag{4.82}$$

and one corresponding to the electron with its spin-down

$$E_1 = E_0 - \left(\frac{e\hbar}{2m_e}\right)B_{so} \tag{4.83}$$

The splitting of the energy state causes a splitting of the spectral line associated with each energy state into two component lines. *This spin-orbit splitting of spectral* *lines is sometimes called the fine structure of the spectral lines.* The variation in wavelength is quite small, of the order of 0.2 nm, which is, however, measurable.

There is no splitting of *s* states because for an *s* state, $l = 0$, which implies that there is no angular momentum. Thus, in the *s* state, the electron does not orbit about the proton in any classical sense. This implies that the proton cannot orbit about the electron and hence, cannot create the magnetic field, $B_{\mathbf{so}}$, at the location of the electron. Therefore, if $B_{\mathbf{so}} = 0$ in equation 4.81, then the potential energy term must also equal zero, and there can be no splitting of such a state.

4.8 The Pauli Exclusion Principle and the Periodic Table of the Elements

An electron in the hydrogen atom can now be completely specified by the four quantum members:

n = the principal quantum number
l = the orbital quantum number
m_l = the orbital magnetic quantum number
m_s = the spin magnetic quantum number

To obtain the remaining chemical elements a building up process occurs. That is, protons and neutrons are added to the nucleus, and electrons are added to the orbits to form the rest of the chemical elements. As an example, the chemical element helium is formed by adding one proton and two neutrons to the nucleus, and one orbital electron to give a total of two electrons, two protons, and two neutrons. The next chemical element, lithium, contains three protons, four neutrons, and three electrons. Beryllium has four protons, five neutrons, and four electrons. In this fashion of adding electrons, protons, and neutrons, the entire table of chemical elements can be generated. But where are these additional electrons located in the atom? Can they all be found in the same orbit? The answer is no, and was stated in the form of the Pauli exclusion principle by the Austrian physicist Wolfgang Pauli (1900-1958) in 1925. *The* ***Pauli exclusion principle*** *states that no two electrons in an atom can exist in the same quantum state.* Because the state of any electron is specified by the quantum numbers n, l, m_l, and m_s, *the exclusion principle states that no two electrons can have the same set of the four quantum numbers.*

Electrons with the same value of n are said to be in the same orbital shell, and the shell designation is shown in table 4.3. Electrons that have the same value of l in a shell are said to occupy the same subshell. Electrons fill up a shell by starting at the lowest energy.

Table 4.3 Atomic Shells					
Quantum number n	1	2	3	4	5
Shell	K	L	M	N	O

The building up of the chemical elements is shown in table 4.4. Thus, the first electron is found in the K shell with quantum numbers (1001/2) and this is the

Table 4.4
Electron States in Terms of the Quantum Numbers

		n	l	m_l	m_s		**Numbers of States for Value of l**	**Total # of States for Value of n**
$n = 1$	$l = 0$	1	0	0	−1/2	$1s$↑	2	2
		1	0	0	−1/2	$1s$↓		
$n = 2$	$l = 0$	2	0	0	1/2	$2s$↑	2	
		2	0	0	−1/2	$2s$↓		
	$l = 1$	2	1	1	1/2			8
		2	1	1	−1/2			
		2	1	0	1/2		6	
		2	1	0	−1/2			
		2	1	−1	1/2			
		2	1	−1	−1/2			
$n = 3$	$l = 0$	3	0	0	1/2	$3s$↑	2	
		3	0	0	−1/2	$3s$↓		
	$l = 1$	3	1	1	1/2			
		3	1	1	−1/2		6	
		3	1	0	1/2			
		3	1	0	−1/2			
		3	1	−1	1/2			
		3	1	−1	−1/2			
	$l = 2$	3	2	2	1/2			18
		3	2	2	−1/2			
		3	2	1	1/2			
		3	2	1	−1/2		10	
		3	2	0	1/2			
		3	2	0	−1/2			
		3	2	−1	1/2			
		3	2	−1	−1/2			
		3	2	−2	1/2			
		3	2	−2	−1/2			

configuration of the hydrogen atom. When the next electron is placed in the atom, it cannot have the quantum numbers of the first electron, so it must now be the electron given by the quantum numbers (100 −1/2), that is, this second electron must have its spin-down. The atom with two electrons is the helium atom, and, as we now see, its two electrons are found in the K shell, one with spin-up the other with spin-down. The addition of the third electron cannot go into the K shell because all of the quantum numbers associated with n = 1 are already used up.

Hence, a third electron must go into the $n = 2$ state or L shell. The rest of the table shows how the process of building up the set of quantum numbers continues. The notation $1s\uparrow$ means that the electron is in the $1s$ state with its spin-up. The notation $1s\downarrow$ means that the electron is in the $1s$ state with its spin-down.

The electron configuration is stated symbolically in the form

$$n(\, l\,)^{\#}$$

where n is the principal quantum number, l is the orbital quantum number expressed in the spectroscopic notation, and # stands for the number of electrons in that subshell. Hence, the electron configuration for the hydrogen atom would be $1s^1$, and for the helium atom, $1s^2$. The electron configuration for the first few chemical elements is shown in table 4.5. Note that the difference between one chemical element and the next is the addition of one more proton and electron.

Table 4.5
Electron Configuration for the First Few Chemical Elements

Chemical Element	Electron Configuration			
H	$1s^1$			
He	$1s^2$			
Li	$1s^2 2s^1$			
Be	$1s^2 2s^2$			
B	$1s^2 2s^2$	$2p^1$		
C	$1s^2 2s^2$	$2p^2$		
N	$1s^2 2s^2$	$2p^3$		
O	$1s^2 2s^2$	$2p^4$		
F	$1s^2 2s^2$	$2p^5$		
Ne	$1s^2 2s^2$	$2p^6$		
Na	$1s^2 2s^2$	$2p^6$	$3s^1$	
Mg	$1s^2 2s^2$	$2p^6$	$3s^2$	
Al			$3s^2$	$3p^1$
Si			$3s^2$	$3p^2$
P			$3s^2$	$3p^3$
S			$3s^2$	$3p^4$
Cl			$3s^2$	$3p^5$
Ar			$3s^2$	$3p^6$

The complete electron configurations for the ground states of all the chemical elements is shown somewhat differently in table 4.6. Thus, the entire set of chemical elements can be built-up in this way.

The way that an element reacts chemically depends on the number of electrons in the outer shell. Hence, all the chemical elements can be grouped into a table that shows how these elements react. Such a table, called the Periodic Table of the Elements and first formulated by the Russian chemist, Dmitri Mendeleev (1834-1907) around 1869, is shown in figure 4.18.

Table 4.6 Electron Configurations for the Ground States of the Elements

		K	L		M			N				O				P			Q
		1*s*	2*s*	2*p*	3*s*	3*p*	3*d*	4*s*	4*p*	4*d*	4*f*	5*s*	5*p*	5*d*	5*f*	6*s*	6*p*	6*d*	7*f*
1	H	1																	
2	He	2																	
3	Li	2	1																
4	Be	2	2																
5	B	2	2	1															
6	C	2	2	2															
7	N	2	2	3															
8	O	2	2	4															
9	F	2	2	5															
10	Ne	2	2	6															
11	Na	2	2	6	1														
12	Mg	2	2	6	2														
13	Al	2	2	6	2	1													
14	Si	2	2	6	2	2													
15	P	2	2	6	2	3													
16	S	2	2	6	2	4													
17	Cl	2	2	6	2	5													
18	Ar	2	2	6	2	6													
19	K	2	2	6	2	6		1											
20	Ca	2	2	6	2	6		2											
21	Sc	2	2	6	2	6	1	2											
22	Ti	2	2	6	2	6	2	2											
23	V	2	2	6	2	6	3	2											
24	Cr	2	2	6	2	6	5	1											
25	Mn	2	2	6	2	6	5	2											
26	Fe	2	2	6	2	6	6	2											
27	Co	2	2	6	2	6	7	2											
28	Ni	2	2	6	2	6	8	2											
29	Cu	2	2	6	2	6	10	1											
30	Zn	2	2	6	2	6	10	2											
31	Ga	2	2	6	2	6	10	2	1										
32	Ge	2	2	6	2	6	10	2	2										
33	As	2	2	6	2	6	10	2	3										
34	Se	2	2	6	2	6	10	2	4										
35	Br	2	2	6	2	6	10	2	5										
36	Kr	2	2	6	2	6	10	2	6										
37	Rb	2	2	6	2	6	10	2	6			1							
38	Sr	2	2	6	2	6	10	2	6			2							
39	Y	2	2	6	2	6	10	2	6	1		2							
40	Zr	2	2	6	2	6	10	2	6	2		2							
41	Nb	2	2	6	2	6	10	2	6	4		1							
42	Mo	2	2	6	2	6	10	2	6	5		1							
43	Tc	2	2	6	2	6	10	2	6	5		2							
44	Ru	2	2	6	2	6	10	2	6	7		1							
45	Rh	2	2	6	2	6	10	2	6	8		1							
46	Pd	2	2	6	2	6	10	2	6	10									
47	Ag	2	2	6	2	6	10	2	6	10		1							
48	Cd	2	2	6	2	6	10	2	6	10		2							

Table 4.6 (Continued)

		K	L			M				N				O			P		Q
		1*s*	2*s*	2*p*	3*s*	3*p*	3*d*	4*s*	4*p*	4*d*	4*f*	5*s*	5*p*	5*d*	5*f*	6*s*	6*p*	6*d*	7*f*
49	In	2	2	6	2	6	10	2	6	10		2	1						
50	Sn	2	2	6	2	6	10	2	6	10		2	2						
51	Sb	2	2	6	2	6	10	2	6	10		2	3						
52	Te	2	2	6	2	6	10	2	6	10		2	4						
53	I	2	2	6	2	6	10	2	6	10		2	5						
54	Xe	2	2	6	2	6	10	2	6	10		2	6						
55	Cs	2	2	6	2	6	10	2	6	10		2	6			1			
56	Ba	2	2	6	2	6	10	2	6	10		2	6			2			
57	La	2	2	6	2	6	10	2	6	10		2	6	1		2			
58	Ce	2	2	6	2	6	10	2	6	10	2	2	6			2			
59	Pr	2	2	6	2	6	10	2	6	10	3	2	6			2			
60	Nd	2	2	6	2	6	10	2	6	10	4	2	6			2			
61	Pm	2	2	6	2	6	10	2	6	10	5	2	6			2			
62	Sm	2	2	6	2	6	10	2	6	10	6	2	6			2			
63	Eu	2	2	6	2	6	10	2	6	10	7	2	6			2			
64	Gd	2	2	6	2	6	10	2	6	10	7	2	6	1		2			
65	Tb	2	2	6	2	6	10	2	6	10	9	2	6			2			
66	Dy	2	2	6	2	6	10	2	6	10	10	2	6			2			
67	Ho	2	2	6	2	6	10	2	6	10	11	2	6			2			
68	Er	2	2	6	2	6	10	2	6	10	12	2	6			2			
69	Tm	2	2	6	2	6	10	2	6	10	13	2	6			2			
70	Yb	2	2	6	2	6	10	2	6	10	14	2	6			2			
71	Lu	2	2	6	2	6	10	2	6	10	14	2	6	1		2			
72	Hf	2	2	6	2	6	10	2	6	10	14	2	6	2		2			
73	Ta	2	2	6	2	6	10	2	6	10	14	2	6	3		2			
74	W	2	2	6	2	6	10	2	6	10	14	2	6	4		2			
75	Re	2	2	6	2	6	10	2	6	10	14	2	6	5		2			
76	Os	2	2	6	2	6	10	2	6	10	14	2	6	6		2			
77	Ir	2	2	6	2	6	10	2	6	10	14	2	6	7		2			
78	Pt	2	2	6	2	6	10	2	6	10	14	2	6	9		1			
79	Au	2	2	6	2	6	10	2	6	10	14	2	6	10		1			
80	Hg	2	2	6	2	6	10	2	6	10	14	2	6	10		2			
81	Tl	2	2	6	2	6	10	2	6	10	14	2	6	10		2	1		
82	Pb	2	2	6	2	6	10	2	6	10	14	2	6	10		2	2		
83	Bi	2	2	6	2	6	10	2	6	10	14	2	6	10		2	3		
84	Po	2	2	6	2	6	10	2	6	10	14	2	6	10		2	4		
85	At	2	2	6	2	6	10	2	6	10	14	2	6	10		2	5		
86	Rn	2	2	6	2	6	10	2	6	10	14	2	6	10		2	6		
87	Fr	2	2	6	2	6	10	2	6	10	14	2	6	10		2	6		1
88	Ra	2	2	6	2	6	10	2	6	10	14	2	6	10		2	6		2
89	Ac	2	2	6	2	6	10	2	6	10	14	2	6	10		2	6	1	2
90	Th	2	2	6	2	6	10	2	6	10	14	2	6	10		2	6	2	2
91	Pa	2	2	6	2	6	10	2	6	10	14	2	6	10	2	2	6	1	2
92	U	2	2	6	2	6	10	2	6	10	14	2	6	10	3	2	6	1	2
93	Np	2	2	6	2	6	10	2	6	10	14	2	6	10	4	2	6	1	2
94	Pu	2	2	6	2	6	10	2	6	10	14	2	6	10	5	2	6	1	2
95	Am	2	2	6	2	6	10	2	6	10	14	2	6	10	6	2	6	1	2
96	Cm	2	2	6	2	6	10	2	6	10	14	2	6	10	7	2	6	1	2
97	Bk	2	2	6	2	6	10	2	6	10	14	2	6	10	8	2	6	1	2
98	Cf	2	2	6	2	6	10	2	6	10	14	2	6	10	10	2	6		2
99	Es	2	2	6	2	6	10	2	6	10	14	2	6	10	11	2	6		2
100	Fm	2	2	6	2	6	10	2	6	10	14	2	6	10	12	2	6		2
101	Md	2	2	6	2	6	10	2	6	10	14	2	6	10	13	2	6		2

Table 4.6 (Continued)																			
		K	L			M			N				O				P		Q
		1*s*	2*s*	2*p*	3*s*	3*p*	3*d*	4*s*	4*p*	4*d*	4*f*	5*s*	5*p*	5*d*	5*f*	6*s*	6*p*	6*d*	7*f*
102	No	2	2	6	2	6	10	2	6	10	14	2	6	10	14	2	6		2
103	Lr	2	2	6	2	6	10	2	6	10	14	2	6	10	14	2	6	1	2
104	Rf																		
105	Ha																		

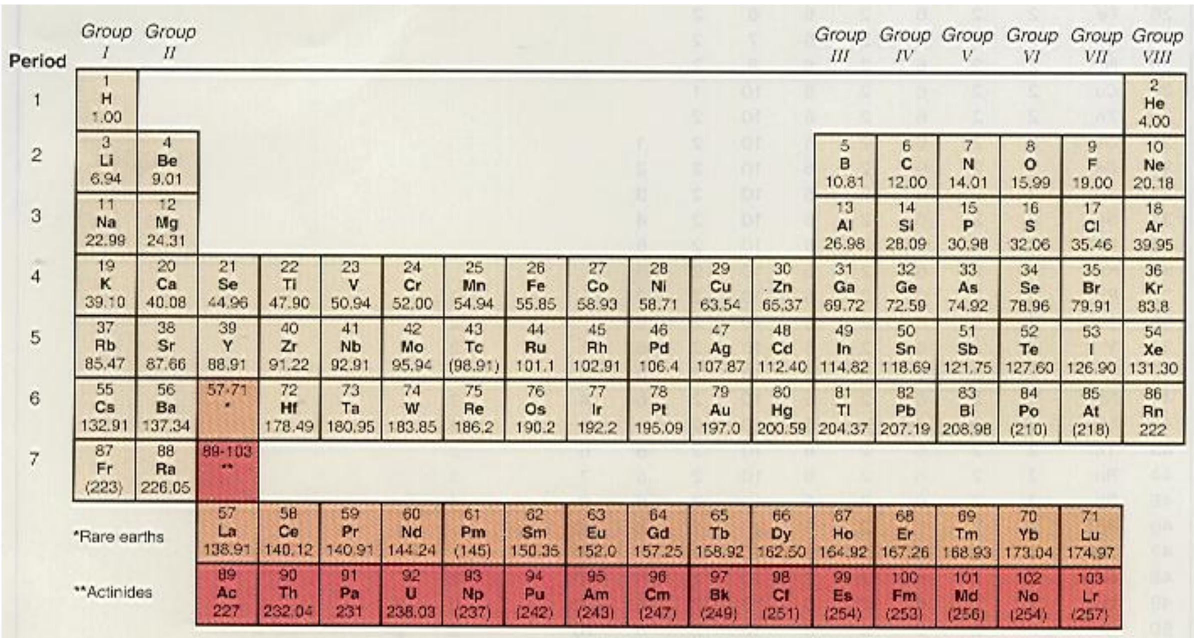

Figure 4.18 Periodic table of the elements.

Notice that there are vertical columns called groups and the chemical elements within each group have very similar properties. As an example, Group I contains elements that have only one electron in their outermost shell and these chemicals react very strongly. The horizontal rows are called periods, and progressing from one column to another in a particular row, the chemical element contains one more electron. Thus in column I, there is one outer electron; in column II, there are two; in column III, there are three; and so on until we get to column VIII, where there are eight electrons in a closed shell. The chemical properties within a period change gradually as the additional electron is added. However, the first element of the period is very active chemically, whereas the last element of a period contains the inert gases. These gases are inert because the outer electron shell is closed and there is no affinity to either gain or lose electrons, and, hence, these elements do not react chemically with any of the other elements. Thus, there is a drastic chemical difference between the elements in Group I and Group VIII. The chemical properties of any element is a function of the number of electrons in the outer shell.

As mentioned earlier, an electron always falls into the state of lowest energy, and this can be seen in table 4.6. For the early elements, each lower quantum state is filled before a higher one starts. However, starting with the element potassium (K), a change occurs in the sequence of quantum numbers. Instead of the 19th electron going into a 3*d* state, it goes into the 4*s* state, as shown in table 4.6. The reason for this is that these elements of higher atomic number start to have an energy dependence on the quantum number *l*, because the higher orbits are partially shielded from the nuclear charge by the inner electrons of low values of *l*. Thus, as *l* increases, the energy of the state also increases. Hence, a 4*s* state is actually at a lower energy than a 3*d* state. Therefore, the 19th electron goes into the $4s^1$ state; the 20th electron goes into the $4s^2$ state; and the 21st electron goes into the 3*d* state, which is lower than the 4*p* state. This is seen in both table 4.6 and figure 4.18. Additional electrons now start to fill up the 3*d* shell as shown. The order in which electron subshells are filled in atoms is given by 1*s*, 2*s*, 2*p*, 3*s*, 3*p*, 4*s*, 3*d*, 4*p*, 5*s*, 4*d*, 5*p*, 6*s*, 4*f*, 5*d*, 6*p*, 7*s*, 6*d*. Table 4.7 is a reproduction of table 4.2 and it can be used to generate the sequence in which electrons fill the orbital subshells by following the diagonal lines traced on the table.

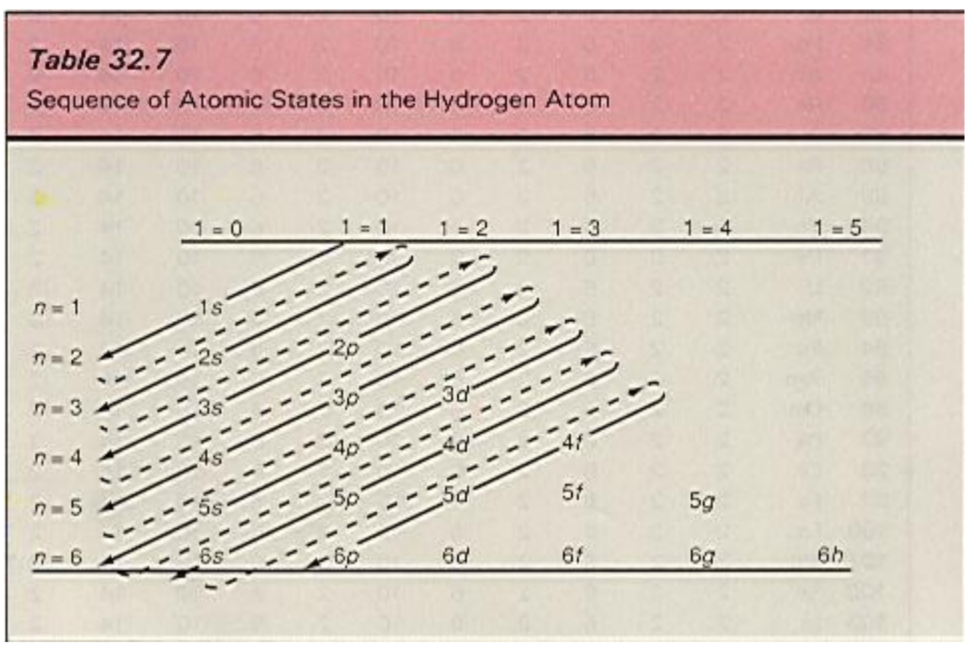

Table 4.7 Sequence of Atomic States in the Hydrogen Atom

One of the characteristics of a closed shell is that the total orbital angular momentum **L** is zero and the total spin angular momentum **S** is also zero. To see this, let us consider the electrons in a closed 2*p* subshell. The angular momentum vectors associated with the (211), (210), and (21–1) quantum numbers are shown in figure 4.19(a). The angular momentum vector associated with the state (211) should be fixed in its direction in space by the requirement that $L_z = \hbar$, as shown in the figure. However, by the *Heisenberg uncertainty principle,* the direction of **L** cannot be so precisely stated. Hence, the angular momentum vector can precess around the

z-axis as shown. Thus, the value of θ is fixed, but **L** precesses around z, always at the same angle θ. Sometimes, **L** is toward the right, sometimes toward the left, sometimes toward the back, and sometimes toward the front. Its mean position is, therefore, in the positive z-direction. The angular momentum vector

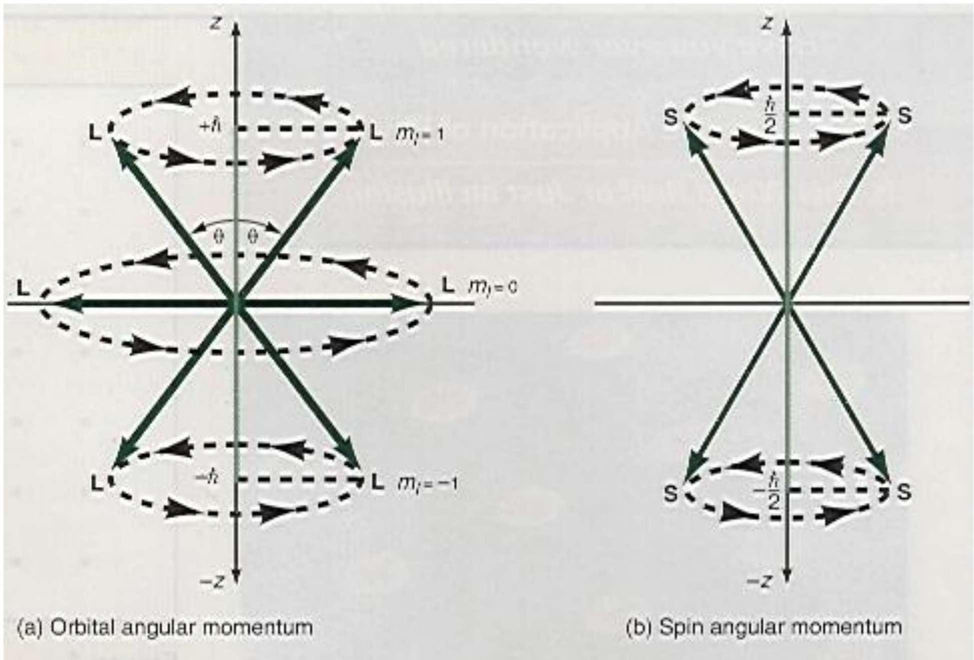

Figure 4.19 The total angular momentum of a closed shell is zero.

associated with (21-1) precesses in the same way about the negative z-axis and its mean position is in the negative z-direction. Since the magnitude of **L** is the same for both vectors, the average or mean value of L for the two states adds up to zero. The mean value of **L** for the (210) state is zero, because sometimes it is toward the right and sometimes toward the left, and so on. Hence, for any closed subshell the total angular momentum **L** is zero. Because the angular momentum vector for the s state is already equal to zero, because $l = 0$, the orbital angular momentum of a completely filled shell is zero.

In the same way, the spin angular momentum **S** also adds up to zero when there are the same number of electrons with spin-up as with spin-down, figure 4.19(b). But this is exactly the case of a closed shell, so the total spin angular momentum of a closed shell is also zero. An atom with a closed shell also has a zero dipole moment because **L** and **S** are both zero.

For the magnetic elements iron (Fe), cobalt (Co), and nickel (Ni), the electrons in the 3d shell are not paired off according to spin. Iron has five electrons with spin-up, cobalt has four, and nickel has three. Thus, the spin angular momentum vectors add up very easily to give a rather large spin magnetic dipole moment. Hence, when a piece of iron is placed in an external magnetic field, all these very strong magnetic dipoles align themselves with the field, thereby producing the ordinary bar magnet.

Have you ever wondered …?
An Essay on the Application of Physics
Is This World Real or Just an Illusion?

Have you ever wondered if this solid world that we see around us is really an illusion? Philosophers have argued this question for centuries. To see for ourselves all we have to do is slam our fist down on the table. Ouch! That table is real, I can tell because my hand hurts where I hit the table. That table is solid and is no illusion.

Let us look a little bit more carefully at the solid table. It certainly looks solid. If we were to take a very powerful microscope and look at the smooth table we would see that the table is made up of a lattice structure, the simplest lattice structure is shown in figure 1. Each one of those dots represents an atom of the

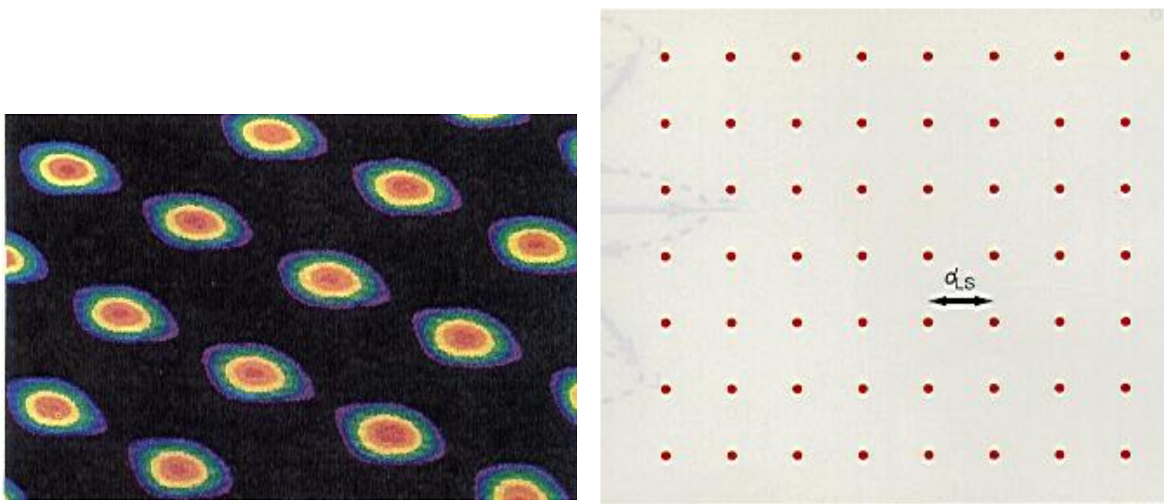

Figure 1 The lattice structure.

material. They are arranged in a symmetric net. The distance separating each atom in the lattice structure is actually quite small, a typical distance is about 5.6×10^{-10} m. The diameter of an atom is about 1.1×10^{-10} m. Hence, the ratio of the separation between atoms in the lattice structure d_{LS} to the diameter of the atom d_A is

$$\frac{d_{LS}}{d_A} = \frac{5.6 \times 10^{-10} \text{ m}}{1.1 \times 10^{-10} \text{ m}} = 5.09 \tag{4H.1}$$

or the distance separating the atoms in the lattice structure is

$$d_{LS} = 5.09\, d_A \tag{4H.2}$$

Equation 4H.2 says that each nearest atom is about 5.09 atomic diameters distant.

If we now look at the problem in three dimensions, the lattice structure looks like a box with the atom at each corner of the box, each separated by the distance d_{LS}. The box is called a *unit cell,* and is shown in figure 2. The volume of the box is given by

$$V_{box} = (d_{LS})^3 = (5.6 \times 10^{-10}\ m)^3 = 1.76 \times 10^{-28}\ m^3 \tag{4H.3}$$

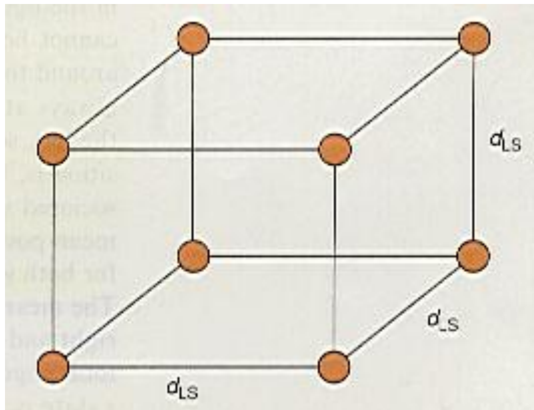

Figure 2 The unit cell.

But part of the volume of each atom is shared with the surrounding boxes. Figure 3 shows four of these boxes where they join. Consider the atom as the sphere located

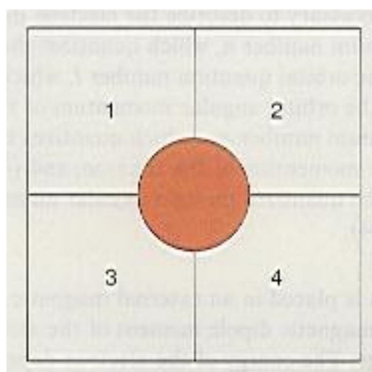

Figure 3 Determining the number of atoms in a unit cell.

at the bottom corner of box 1 and protruding into boxes 2, 3, and 4. Each box shown contains ¼ of the volume of the sphere. There are four more boxes in front of the four boxes shown here. Hence, each box contains 1/8 of the volume of the sphere. Therefore, each box contains the atomic volume of

$$\left(\frac{8\ \text{atoms}}{\text{box}}\right)\left(\frac{1}{8}\right)\left(\frac{\text{atomic volume}}{\text{atom}}\right) = \frac{1\ \text{atomic volume}}{\text{box}} \tag{4H.4}$$

That is, the unit cell or box contains the equivalent of one atom. The volume occupied by the atom in the box is

$$V_{\text{atomic}} = \frac{4}{3}\pi r^3 \tag{4H.5}$$
$$= \frac{4}{3}\pi(0.55 \times 10^{-10} \text{ m})^3$$
$$= 6.95 \times 10^{-31} \text{ m}^3$$

The ratio of the volume of the box to the volume of the atom contained in the box is

$$\frac{V_{\text{box}}}{V_{\text{atomic}}} = \frac{1.76 \times 10^{-28} \text{ m}^3}{6.95 \times 10^{-31} \text{ m}^3} = 2.53 \times 10^2$$

Hence, the volume of the box is

$$V_{\mathbf{box}} = 253\ V_{\text{atomic}} \tag{4H.6}$$

That is, the volume of one unit cell of the lattice structure is 253 times the volume occupied by the atom. Hence, the solid table that you see before you, constructed from that lattice structure, is made up of a great deal of empty space.

The atoms making up the lattice structure are also composed of almost all empty space. For example, the simplest atom, hydrogen, shown in figure 4, has a diameter $d_{\mathbf{A}}$ of about 1.1×10^{-10} m. The diameter of the nucleus is about 1×10^{-14} m.

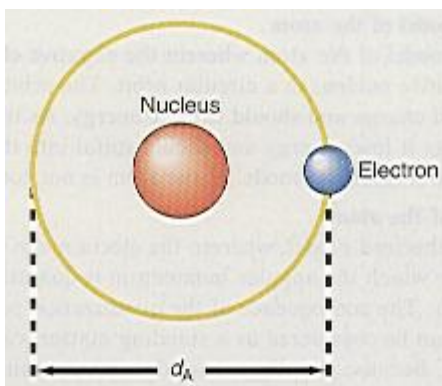

Figure 4 The size of an atom.

Because the mass of the electron is so small compared to the mass of the proton and neutron, the nucleus contains about 99.9% of the mass of the atom. The size of the electron is so small that its volume can be neglected compared to the volume of the nucleus and the atom. The ratio of the volume of the atom to the volume of the nucleus is about

$$\frac{V_{\text{A}}}{V_{\text{N}}} = \frac{\frac{4}{3}\pi r_{\text{A}}^3}{\frac{4}{3}\pi r_{\text{N}}^3} = \frac{r_{\text{A}}^3}{r_{\text{N}}^3} \tag{4H.7}$$
$$= \frac{(0.55 \times 10^{-10} \text{ m})^3}{(0.5 \times 10^{-14} \text{ m})^3} = 1.1 \times 10^{12}$$

Hence, the volume of the atom is

$$V_A = 1.1 \times 10^{12}\ V_N = 1{,}100{,}000{,}000{,}000\ V_N \tag{4H.8}$$

That is, the volume of the atom is over one trillion times the volume of the nucleus. Hence, the atoms that make up that lattice structure are also composed of almost all empty space. When we combine the volume of the box (unit cell) with respect to the volume of each atom in the box, equation 4H.6, with the volume of the atom with respect to the nucleus, equation 4H.8, we get

$$V_{\mathbf{box}} = 253\ V_{\text{atomic}} = 253(1.1 \times 10^{12}\ V_{\mathrm{N}}) = 278 \times 10^{12}\ V_{\mathrm{N}}$$

or *the volume of the box (unit cell) is 278 trillion times the volume of the nucleus. Therefore, the solid is made up almost entirely of empty space.*

But if a solid consists almost entirely of empty space, then why can't you put your hand through the solid? You can't place your hand through the solid because there are electrical and atomic forces that hold the atom and lattice structure together, and your hand cannot penetrate that force field.

You can't put your hand through a block of ice either, but by heating the ice you give energy to the water molecules that make up the ice, and that energy is enough to pull the molecules away from the lattice structure, thereby melting the ice. You can now put your hand in the water, even though you could not put it through the ice. If you heat the water further, the water evaporates into the air and becomes invisible. You can walk through the air containing the water vapor as though it weren't even there.

So, is the world real or only an illusion? The world is certainly real because it is made up of all those atoms and molecules. But is it an illusion? In the sense described here, yes it is. But it is truly a magnificent illusion. For this solid world that we live in is composed almost entirely of empty space. It is a beautiful stage on which we all act out our lives.

The Language of Physics

Rutherford model of the atom
A planetary model of the atom wherein the negative electron orbits about the positive nucleus in a circular orbit. The orbiting electron is an accelerated charge and should radiate energy. As the electron radiates energy it loses energy and should spiral into the nucleus. Therefore, the Rutherford model of the atom is not correct (p.).

Bohr theory of the atom
A revised Rutherford model, wherein the electron can be found only in an orbit for which the angular momentum is quantized in multiples of $\hbar$. The consequence of the quantization postulate is that the electron can be considered as a standing matter wave in the electron orbit. Because standing waves do not transmit energy, the electron does not radiate energy while in its orbit and does not spiral into the nucleus. The Bohr model is thus stable. Bohr then postulated that when the electron jumps from a higher energy orbit to a lower energy orbit, a photon of light is emitted. Thus, the spectral lines of the hydrogen atom should be discrete, agreeing with experimental results. However, the Bohr theory could not explain the spectra from multielectron atoms and it is not, therefore, a completely accurate model of the atom (p.).

Quantum mechanical model of the atom
This model arises from the application of the Schrödinger equation to the atom. The model says that the following four quantum numbers are necessary to describe the electron in the atom: (1) the principal quantum number n, which quantizes the energy of the electron; (2) the orbital quantum number l, which quantizes the magnitude of the orbital angular momentum of the electron; (3) the magnetic quantum number m_l, which quantizes the direction of the orbital angular momentum of the electron; and (4) the spin quantum number s, which quantizes the spin angular momentum of the electron (p.).

Zeeman effect
When an atom is placed in an external magnetic field a torque acts on the orbital magnetic dipole moment of the atom giving it a potential energy. The energy of the electron depends on the magnetic quantum number as well as the principal quantum number. For a particular value of n, there are multiple values of the energy. Hence, instead of a single spectral line associated with a transition from the n^{th} state to the ground state, there are many spectral lines depending on the value of m_l. Thus, a single spectral line has been split into several spectral lines (p.).

Pauli exclusion principle
No two electrons in an atom can exist in the same quantum state. Hence, no two electrons can have the same quantum numbers (p.).

Summary of Important Equations

Distance of closest approach to nucleus $$r_0 = \frac{2kZe^2}{\text{KE}} \tag{4.7}$$

Relative size of atom $$r_a = 10{,}000\ r_n \tag{4.8}$$

Radius of nucleus $$R = R_0A^{1/3} \tag{4.9}$$

Bohr Theory of the Hydrogen Atom

Angular momentum is quantized $$L = mvr = n\hbar \tag{4.25}$$

Orbital radius $$r_n = \frac{\hbar^2}{kme^2} n^2 \tag{4.31}$$

$$r_n = r_1 n^2 \tag{4.32}$$

Orbital velocity $$v_n = \frac{ke^2}{n\hbar} \tag{4.33}$$

$$v_n = \frac{v_1}{n} \tag{4.34}$$

Electron energy $$E_n = -\frac{mk^2e^4}{2n^2\hbar^2} \tag{4.35}$$

$$E_n = -\frac{E_1}{n^2} \tag{4.37}$$

Einstein's relation $$h\nu = E_i - E_f \tag{4.38}$$

Frequency of emitted photon $$\nu = \frac{E_1}{h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{4.39}$$

Wavelength of emitted photon $$\frac{1}{\lambda} = \frac{E_1}{hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{4.40}$$

Quantum Mechanical Theory of the Hydrogen Atom

Principal quantum number $n = 1, 2, 3, \ldots$ (4.42)

Orbital quantum number $l = 0, 1, 2, \ldots, (n-1)$ (4.43)

Magnetic quantum number $m_l = 0, \pm 1, \pm 2, \ldots, \pm l$ (4.44)

Electron energy $$E_n = -\frac{k^2e^4m}{2\hbar^2}\frac{1}{n^2} \tag{4.45}$$

Angular momentum $$L = \sqrt{l(l+1)}\ \hbar \tag{4.46}$$

z-component of angular momentum $$L_z = m_l\hbar \tag{4.47}$$

Direction of **L** $$\theta = \cos^{-1}\frac{m_l}{\sqrt{l(l+1)}} \tag{4.49}$$

Orbital magnetic dipole moment

$$\boldsymbol{\mu}_l = -\frac{e}{2m_e}\mathbf{L} \tag{4.57}$$

$$\mu_l = \frac{e\hbar}{2m_e}\sqrt{l(l+1)} \tag{4.58}$$

Potential energy of a dipole in an external magnetic field

$$\text{PE} = \mu_l B \cos\theta \tag{4.59}$$

$$\text{PE} = m_l \frac{e\hbar}{2m_e}\text{B} \tag{4.62}$$

Zeeman Effect
Splitting of energy state

$$E' = E + m_l \frac{(e\hbar)B}{2m_e} \tag{4.66}$$

Splitting of spectral lines in an external magnetic field

$$\lambda_+ = \frac{c}{\nu_+}$$

$$\lambda_+ = \frac{c}{\nu_0 + e\hbar B/2m_e h} \tag{4.70}$$

$$\lambda = \frac{c}{\nu_0} = \frac{ch}{\frac{3}{4}E_1} \tag{4.65}$$

$$\lambda_- = \frac{c}{\nu_-}$$

$$\lambda_- = \frac{c}{\nu_0 - e\hbar B/2m_e h} \tag{4.71}$$

Selection rules for transitions

$$\begin{aligned} \Delta l &= \pm 1 \\ \Delta m_l &= 0, \ \pm 1 \end{aligned} \tag{4.72}$$

Spin quantum number $$s = \frac{1}{2} \tag{4.73}$$

Spin angular momentum $$S = \sqrt{s(s+1)}\ \hbar \tag{4.74}$$

$$S = \tfrac{1}{2}\sqrt{3}\ \hbar \tag{4.75}$$

z-component of spin $$S_z = m_s\hbar \tag{4.76}$$

$$S_z = \pm\,\hbar/2 \tag{4.77}$$

Spin magnetic dipole moment $$\mu_s = -2.0024\left(\frac{e}{2m_e}\right)S \tag{4.78}$$

Potential energy of electron due to spin

$$\mathrm{PE} = -\mu_s B \cos\theta \tag{4.79}$$

$$\mathrm{PE} = +2\left(\frac{e}{2m_e}\right) SB\cos\theta \tag{4.80}$$

$$\mathrm{PE} = \pm\left(\frac{e\hbar}{2m_e}\right) B_{so} \tag{4.81}$$

Spin-orbit splitting of energy state

$$E_1 = E_0 + \left(\frac{e\hbar}{2m_e}\right) B_{so} \tag{4.82}$$

$$E_1 = E_0 - \left(\frac{e\hbar}{2m_e}\right) B_{so} \tag{4.83}$$

Questions for Chapter 4

1. Discuss the differences among (a) the plum pudding model of the atom, (b) the Rutherford model of the atom, (c) the Bohr theory of the atom, and (d) the quantum mechanical theory of the atom.

*2. Discuss the effect of the uncertainty principle and the Bohr theory of electron orbits.

*3. How can you use spectral lines to determine the chemical composition of a substance?

4. If you send white light through a prism and then send it through a tube of hot hydrogen gas, what would you expect the spectrum to look like when it emerges from the hydrogen gas?

*5. In most chemical reactions, why are the outer electrons the ones that get involved in the reaction? Is it possible to get the inner electrons of an atom involved?

6. When an atom emits a photon of light what does this do to the angular momentum of the atom?

7. Explain how the Bohr theory can be used to explain the spectra from singly ionized atoms.

*8. Discuss the process of absorption of light by matter in terms of the atomic structure of the absorbing medium.

*9. Rutherford used the principle of scattering to “see” inside the atom. Is it possible to use the principle of scattering to “see” inside a proton and a neutron?

*10. How can you determine the chemical composition of a star?

*11. How can you determine if a star is approaching or receding from you? How can you determine if it is a small star or a very massive star?

Problems for Chapter 4

Section 4.1 The History of the Atom

1. Find the potential energy of two α particles when they are brought together to a distance of 1.20×10^{-15} m, the approximate size of a nucleus.

2. Find the potential energy of an electron and a proton when they are brought together to a distance of 5.29×10^{-11} m to form a hydrogen atom.

3. A silver nucleus is bombarded with 8-MeV α particles. Find (a) the maximum radius of the silver nucleus and (b) the more probable radius of the silver nucleus.

4. Estimate the radius of a nucleus of $^{238}_{92}U$.

5. How many electrons, protons, and neutrons are there in a silver atom?

6. How many electrons, protons, and neutrons are there in a uranium atom?

7. Find the difference in the orbital radius of an electron in the Rutherford atom if the electron is initially in an orbit of 5.29×10^{-11} m radius and the atom radiates 2.00 eV of energy.

Section 4.2 The Bohr Theory of the Atom

*8. An electron is in the third Bohr orbit. Find (a) the radius, (b) the speed, (c) the energy, and (d) the angular momentum of the electron in this orbit.

9. The orbital electron of a hydrogen atom moves with a speed of 5.459×10^5 m/s. (a) Determine the value of the quantum number n associated with this electron. (b) Find the radius of this orbit. (c) Find the energy of the electron in this orbit.

*10. An electron in the third Bohr orbit drops to the ground state. Find the angular momentum of the electron in (a) the third Bohr orbit, and (b) the ground state. (c) Find the change in the angular momentum of the electron. (d) Where did the angular momentum go?

11. At what temperature will the average thermal speed of a free electron equal the speed of an electron in its second Bohr orbit?

12. The lifetime of an electron in an excited state of an atom is about 10^{-8} s. How many orbits will an electron in the $2p$ state execute before falling back to the ground state?

13. Show that the ratio of the speed of an electron in the first Bohr orbit to the speed of light is equal to 1/137. This ratio is called the *fine-structure constant*.

14. Find the radius of the first Bohr orbit of an electron in a singly ionized helium atom.

15. Find the angular momentum of an electron in the third Bohr orbit and the second Bohr orbit. How much angular momentum is lost when the electron drops from the third orbit to the second orbit?

Section 4.3 The Bohr Theory and Atomic Spectra

16. An electron in the third Bohr orbit drops to the second Bohr orbit. Find (a) the energy of the photon emitted, (b) its frequency, and (c) its wavelength.

17. An electron in the third Bohr orbit drops to the ground state. Find (a) the energy of the photon emitted, (b) its frequency, and (c) its wavelength.

18. Calculate the wavelength of the first two lines of the Paschen series.

Section 4.4 The Quantum Mechanical Model of the Hydrogen Atom

19. Find the angular momentum of the electron in the quantum mechanical model of the hydrogen atom when it is in the $2p$ state. How much angular momentum is lost when the electron drops to the $1s$ state?

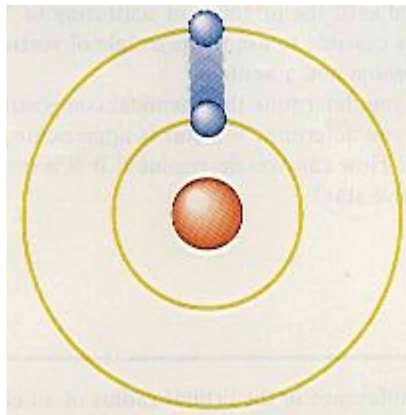

Diagram for problem 19.

20. Find the angular momentum of the electron in the quantum mechanical model of the hydrogen atom when it is in the $3d$ state. (a) How much angular momentum is lost when the electron drops to the $1s$ state? (b) How much energy is lost when the electron drops to the $1s$ state?

21. An orbital electron is in the $5d$ state. (a) Find the energy of the electron in this state. (b) Find the orbital angular momentum of the electron. (c) Compute all possible values of the z-component of the orbital angular momentum of the electron. (d) Determine the largest value of θ, the angle between the orbital angular momentum vector and the z-axis.

22. Find the angle θ that the angular momentum vector makes with the z-axis when the electron is in the $3d$ state.

23. Find (a) the z-component of the angular momentum of an electron when it is in the $2p$ state and (b) the angle **L** makes with the z-axis.

Section 4.5 The Magnetic Moment of the Hydrogen Atom

24. Find the orbital magnetic dipole moment of an electron in the hydrogen atom when it is in the $4f$ state.

25. Find the torque acting on the magnetic dipole of an electron in the hydrogen atom when it is in the $3d$ state and is in an external magnetic field of 2.50×10^{-3} T.

26. Find the additional potential energy of an electron in a $2p$ state when the atom is placed in an external magnetic field of 2.00 T.

Section 4.6 The Zeeman Effect

*27. Find (a) the total energy of an electron in the three $2p$ states when it is placed in an external magnetic field of 2.00 T, (b) the energy of the photons emitted when the electrons fall to the ground state, (c) the frequencies of the spectral lines associated with these transitions, and (d) the wavelengths of their spectral lines.

Section 4.7 Electron Spin

28. Calculate the magnitude of the spin magnetic dipole moment of the electron in a hydrogen atom.

29. Find the potential energy associated with the spin magnetic dipole moment of the electron in a hydrogen atom when it is placed in an external magnetic field of 2.50×10^{-3} T.

Section 4.8 The Pauli Exclusion Principle and the Periodic Table of the Elements

30. How many electrons are necessary to fill the N shell of an atom?

31. Write the electron configuration for the chemical element potassium.

32. Write the electron configuration for the chemical element iron.

33. Enumerate the quantum states (n, l, m_l, m_s) of each of the orbital electrons in the element $^{40}_{20}Ca$.

Additional Problems

34. Find the velocity of an electron in a 5.29×10^{-11} m radius orbit by (a) the Rutherford model and (b) the Bohr model of the hydrogen atom.

35. In what quantum state must an orbital electron be such that its orbital angular momentum is 4.719×10^{-34} J s (i.e., find l, the orbital angular momentum quantum number).

*4. Determine the angular momentum of the moon about the earth. (a) Use Bohr's postulate of quantization of angular momentum and determine the quantum number associated with this orbit. (b) If the quantum number n increases by 1, what is the new angular momentum of the moon? (c) What is the change in the orbital radius of the moon for this change in the quantum number? (d) Is it reasonable to neglect quantization of angular momentum for classical orbits?

37. From the frame of reference of the electron in the hydrogen atom, the proton is in an orbit about the electron and constitutes a current loop. Determine the magnitude of the magnetic field produced by the proton when the electron is in the $2p$ state.

*38. Using the results of problem 37, (a) determine the additional potential energy of an electron caused by the spin-orbit interaction. (b) Find the change in energy when electrons drop from the $2p$ state back to the ground state. (c) Find the frequencies of the emitted photons. (d) Find the wavelengths of the emitted photons.

Interactive Tutorials

39. *Bohr theory of the atom.* An electron is in a Bohr orbit with a principal quantum number $n_i = 3$, and then jumps to a final orbit for the final value $n_f = 1$, find (a) the radius of the n^{th} orbit, (b) the speed of the electron in the n^{th} orbit, (c) the energy of the electron in the initial n_i orbit, (d) the energy of the electron in the final n_f orbit, (e) the energy given up by the electron as it jumps to the lower orbit, (f) the frequency, and (g) the wavelength of the spectral line associated with the transition from the initial $n_i = 3$ state to the final $n_f = 1$ state.

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Chapter 5 Nuclear Physics

"Some 15 years ago the radiation of uranium was discovered by Henri Becquerel and two years later the study of this phenomenon was extended to other substances, first by me, and then by Pierre Curie and myself. This study rapidly led us to the discovery of new elements, the radiation of which, while being analogous with that of uranium, was far more intense. All the elements emitting such radiation I have termed radioactive, and the new property of matter revealed in this emission has thus received the name radioactivity." Marie Curie, 1911

5.1 Introduction

In 1896, Henri Becquerel (1852-1908) found that an ore containing uranium emits an invisible radiation that can penetrate paper and expose a photographic plate. After Becquerel's discovery, Marie (1867-1934) and Pierre (1859-1906) Curie discovered two new radioactive elements that they called polonium and radium. The Curies performed many experiments on these new elements and found that their radioactivity was unaffected by any physical or chemical process. As seen in chapter 4, chemical effects are caused by the interaction with atomic electrons. The reason for the lack of chemical changes affecting the radioactivity implied that radioactivity has nothing to do with the orbital electrons. Hence, the radioactivity must come from within the nucleus.

Rutherford investigated this invisible radiation from the atomic nucleus by letting it move in a magnetic field that is perpendicular to the paper, as shown in figure 5.1. Some of the particles were bent upward, some downward, while others

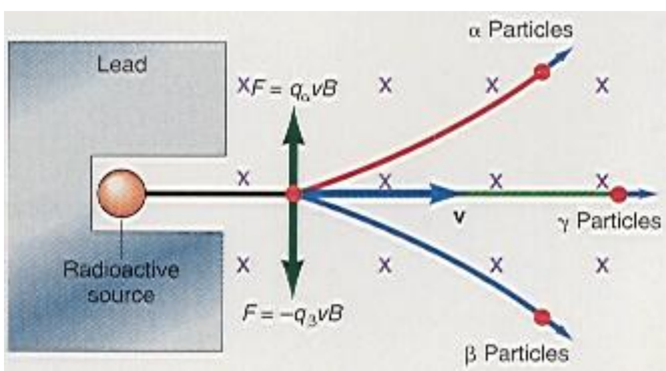

Figure 5.1 Radioactive particles.

went straight through the magnetic field without being bent at all. The particles that were bent upward were called alpha particles, α*;* those bent downward, beta particles, β; and those that were not deviated, gamma particles, γ.

We saw in general physics that the magnitude of the force that acts on a particle of charge q moving at a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$F = qvB \sin\theta$$

(Recall that the direction of the magnetic force is found by the right-hand rule. Place your right hand in the direction of the velocity vector **v.** On rotating your hand toward the magnetic field **B,** your thumb points in the direction of the force **F** acting on the particle.) If the charge of the α particle is positive, then the force acts upward in figure 5.1, and the α particle should be deflected upward. Because the α particle is observed to move upward, its charge must indeed be positive. Its magnitude was found to be twice that of the electronic charge. (Later the α particle was found to be the nucleus of the helium atom.)

Because the β particle is deflected downward in the magnetic field, it must have a negative charge. (The β particles were found to be high energy electrons.) The fact that the γ particle was not deflected in the magnetic field indicated that the γ particle contained no electric charge. The γ particles have since been found to be very energetic photons.

The energies of the α, β, and γ particles are of the order of 0.1 MeV up to 10 MeV, whereas energies of the orbital electrons are of the order of electron volts. Also, the α particles were found to be barely able to penetrate a piece of paper, whereas β particles could penetrate a few millimeters of aluminum, and the γ rays could penetrate several centimeters of lead. Hence, these high energies were further evidence to support the idea that these energetic particles must be coming from the nucleus itself.

5.2 Nuclear Structure

After quantum mechanics successfully explained the properties of the atom, the next questions asked were, What is the nature and structure of the nucleus? How are the protons and neutrons arranged in the nucleus? Why doesn't the nucleus blow itself apart by the repulsive force of the protons? If β particles that come out of a nucleus are electrons, are there electrons in the nucleus? We will discuss these questions shortly.

As seen in chapter 4, the nucleus is composed of protons and neutrons. These protons and neutrons are collectively called *nucleons. The number of protons in the nucleus is given by the* ***atomic number Z,*** *whereas the mass number A is equal to the number of protons plus neutrons in the nucleus. The number of neutrons in a nucleus is given by the* ***neutron number N,*** *which is just the difference between the mass number and the atomic number, that is,*

$$N = A - Z \tag{5.1}$$

A nucleus is represented symbolically in the form

$$_{Z}^{A}X \tag{5.2}$$

with the mass number A displayed as a superscript and the atomic number Z displayed as a subscript and where X is the nucleus of the chemical element that is given by the atomic number Z. As an example, the notation

$$^{12}_{6}C$$

represents the nucleus of the carbon atom that has an atomic number of 6 indicating that it has 6 protons, while the 12 is the mass number indicating that there are 12 nucleons in the nucleus. The number of neutrons, given by equation 5.1, is

$$\begin{aligned} N &= A - Z \\ &= 12 - 6 \\ &= 6 \end{aligned}$$

Every chemical element is found to have isotopes. *An* ***isotope*** *of a chemical element has the same number of protons as the element but a different number of neutrons than the element.* Hence, an isotope of a chemical element has the same atomic number Z but a different mass number A and a different neutron number N. Since the chemical properties of an element are determined by the number of orbiting electrons, an isotope also has the same number of electrons and hence reacts chemically in the same way as the parent element. Its only observable difference chemically is its different atomic mass, which comes from the excess or deficiency of neutrons in the nucleus.

An example of an isotope is the carbon isotope

$$^{14}_{6}C$$

which has the same 6 protons as the parent element but now has 14 nucleons, indicating that there are now 14 – 6 = 8 neutrons. The simplest element, hydrogen, has two isotopes, so there are three types of hydrogen:

$^{1}_{1}\mathrm{H}$ — Normal hydrogen contains 1 proton and 0 neutrons

$^{2}_{1}\mathrm{H}$ — Deuterium contains 1 proton and 1 neutron

$^{3}_{1}\mathrm{H}$ — Tritium contains 1 proton and 2 neutrons

Most elements have two or more stable isotopes. Hence, any chemical sample usually contains isotopes. The **atomic mass** of an element is really an average of the masses of the different isotopes. The abundance of isotopes of a particular element is usually quite small. For example, deuterium has an abundance of only 0.015%. Hence, the actual atomic mass is very close to the mass number A. There are a few exceptions to this, however, one being the chemical element chlorine. As seen from the table of the elements, the atomic mass of chlorine is 35.5, rounded to three significant figures. Contained in that chlorine sample is ${}^{35}_{17}Cl$ and ${}^{37}_{17}Cl$ The abundance of ${}^{35}_{17}Cl$ is 75.5%, whereas the abundance of ${}^{37}_{17}Cl$ is 24.5%. The atomic mass of chlorine is the average of these two forms of chlorine, weighted by the amount of each present in a sample. Thus, the atomic mass of chlorine is

$$\text{Atomic mass} = 35(0.755) + 37(0.245) = 35.5$$

In general, the atomic mass of any element is

$$\text{Atomic mass} = A_1(\text{\% Abundance}) + A_2(\text{\% Abundance}) + A_3(\text{\% Abundance}) +$$

where A_1, A_2, and A_3, is the mass number of a particular isotope.

In chapters 3 and 4, the masses of the proton and neutron are given as

$$m_p = 1.6726 \times 10^{-27}\ \text{kg} = 1.00726\ \text{u} = 938.256\ \text{MeV}$$
$$m_n = 1.6749 \times 10^{-27}\ \text{kg} = 1.00865\ \text{u} = 939.550\ \text{MeV}$$

The protons in a nucleus are charged positively, thus Coulomb's law mandates a force of repulsion between these protons and the nucleus should blow itself apart. Because the nucleus does not blow itself apart, we conclude that there must be another force within the nucleus holding these protons together. *This nuclear force is called the **strong nuclear force** or the strong interaction. The strong force acts not only on protons but also on neutrons and is thus the force that binds the nucleus together.* The strong force has a very short range. That is, it acts within a distance of approximately 10^{-14} m, the order of the size of the nucleus. Outside the nucleus, there is no trace whatsoever of this force. The strong nuclear force is the strongest force known.

If we plot the number of neutrons in a nucleus N against the number of protons in that same nucleus Z for several nuclei, we obtain a graph similar to the one in figure 5.2. For light nuclei the number of neutrons is approximately equal

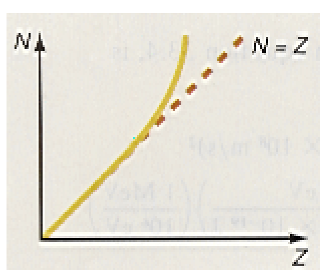

Figure 5.2 Plot of N vs. Z for atomic nuclei.

to the number of protons, as seen by the line labeled $N = Z$. As the atomic number Z increases there are more neutrons in the nucleus than there are protons. Recall that the electrostatic repulsive force acts only between the protons, while the strong nuclear force of attraction acts between the protons and the neutrons. Hence, the additional neutrons increase the attractive force without increasing the repulsive electric force and, thereby, add to the stability of the nucleus. Whenever the nuclear force of attraction is greater than the electrostatic force of repulsion, the nucleus is stable. *Whenever the nuclear force is less than the electrostatic force, the nucleus breaks up or decays, and emits radioactive particles.* The chemical elements with atomic numbers Z greater than 83 have unstable nuclei and decay.

The internal structure of the nucleus is determined in much the same way as in Rutherford scattering. The nucleus is bombarded by high energy electrons (several hundred mega electron volts), that penetrate the nucleus and react electrically with the protons within the nucleus. The results of such scattering experiments seem to indicate that the protons and neutrons are distributed rather evenly throughout the nucleus, and the nucleus itself is generally spherical or ellipsoidal in shape.

It is usually assumed that the whole is always equal to the sum of its parts. This is not so in the nucleus. *The results of experiments on the masses of different nuclei shows that the mass of the nucleus is always less than the total mass of all the protons and neutrons making up the nucleus.* In the nucleus, the missing mass is called the **mass defect,** Δm, given by

$$\Delta m = Zm_{\mathrm{p}} + (A - Z)m_{\mathrm{n}} - m_{\mathrm{nucleus}} \tag{5.3}$$

Because Z is the total number of protons, and $m_{\mathbf{p}}$ is the mass of a proton, $Zm_{\mathbf{p}}$ is the total mass of all the protons. As shown in equation 5.1, $A - Z$ is the total number of neutrons, and since $m_{\mathbf{n}}$ is the mass of a single neutron, $(A - Z)m_{\mathbf{n}}$ is the total mass of all the neutrons. The term m_{nucleus} is the experimentally measured mass of the entire nucleus. Hence, equation 5.3 represents the difference in mass between the sum of the masses of its constituents and the mass of the nucleus itself.

The missing mass is converted to energy in the formation of the nucleus. This energy is found from Einstein's mass-energy relation,

$$E = (\Delta m)c^2 \tag{5.4}$$

and is called the **binding energy** (BE) of the nucleus. From equations 5.3 and 5.4, the binding energy of a nucleus is

$$\mathrm{BE} = (\Delta m)c^2 = Zm_p c^2 + (A - Z)m_n c^2 - m_{nucleus}c^2 \tag{5.5}$$

Example 5.1

The mass defect and the binding energy of the deuteron. Find the mass defect and the binding energy of the deuteron nucleus. The experimental mass of the deuteron is 3.3435×10^{-27} kg.

Solution

The mass defect for the deuteron is found from equation 5.3, with

$$\begin{aligned}\Delta m &= m_{\mathbf{p}} + m_{\mathbf{n}} - m_{\mathrm{D}} \\ &= 1.6726 \times 10^{-27}\ \mathrm{kg} + 1.6749 \times 10^{-27}\ \mathrm{kg} - 3.3435 \times 10^{-27}\ \mathrm{kg} \\ &= 4.00 \times 10^{-30}\ \mathrm{kg}\end{aligned}$$

The binding energy of the deuteron, found from equation 5.4, is

$$\begin{aligned}\mathrm{BE} &= (\Delta m)c^2 \\ &= (4.00 \times 10^{-30}\ \mathrm{kg})(2.9979 \times 10^{8}\ \mathrm{m/s})^2 \\ &= (3.5950 \times 10^{-13}\ \mathrm{J})\left(\frac{1\ \mathrm{eV}}{1.60218 \times 10^{-19}\ \mathrm{J}}\right)\left(\frac{1\ \mathrm{MeV}}{10^6 \mathrm{eV}}\right) \\ &= 2.24\ \mathrm{MeV}\end{aligned}$$

Therefore, the bound constituents have less energy than when they are free. That is, the binding energy comes from the mass that is lost in the process of formation. Conversely, an amount of energy equal to the binding energy is the amount of energy that must be supplied to a nucleus if the nucleus is to be broken up into protons and neutrons. Thus, the binding energy of a nucleus is similar to the ionization energy of an electron in the atom.

Go to Interactive Example

5.3 Radioactive Decay Law

The spontaneous emission of radiation from the nucleus of an atom is called ***radioactivity.*** Radioactivity is the result of the decay or disintegration of unstable nuclei. Radioactivity occurs naturally from all the chemical elements with atomic numbers greater than 83, and can occur naturally from some of the isotopes of the chemical elements below atomic number 83. Some can also occur artificially from nearly all of the chemical elements.

The rate of radioactive emission is measured by the *radioactive decay law.* The number of nuclei dN that disintegrate during a particular time interval dt is directly proportional to the number of nuclei N present. That is

$$\frac{dN}{dt} \propto -N$$

To make an equality of this, we introduce the constant of proportionality λ, called the *decay constant* or *disintegration constant,* and we obtain

$$\frac{dN}{dt} = -\lambda N \tag{5.6}$$

The minus sign in equation 5.6 is necessary because the final number of nuclei N_f is always less than the initial number of nuclei N_i; hence $dN = N_f - N_i$ is always a negative quantity because there is always less radioactive nuclei with time. The decay constant λ is a function of the particular isotope of the chemical element. A large value of λ indicates a large decay rate, whereas a small value of λ indicates a small decay rate. The quantity $-dN/dt$ in equation 5.6 is the rate at which the nuclei decay with time and it is also called the **activity** and designated by the symbol A. Hence,

$$A = -\frac{dN}{dt} = \lambda N \tag{5.7}$$

We must be careful in what follows not to confuse the symbol A for activity with the same symbol A for mass number. It should always be clear in the particular context used.

The SI unit of activity is the becquerel where 1 becquerel (Bq) is equal to one decay per second. That is,

$$1 \text{ Bq} = 1 \text{ decay/s}$$

An older unit of activity, the curie, abbreviated Ci, is equivalent to

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ Bq}$$

Smaller units of activity are the millicurie (10^{-3} curie = mCi) and the microcurie (10^{-6} curie = μCi).

The total number of nuclei present at any instant of time is found by integrating equation 5.6, from t = 0 to the time t as

$$\frac{dN}{dt} = -\lambda N$$
$$\int_{N_0}^{N} \frac{dN}{N} = -\int_0^t \lambda dt$$

Notice that when $t = 0$, the lower limit on the right-hand side of the equation, the initial number of nuclei present is N_0, the lower limit on the left-hand side of the equation, and when $t = t$, the upper limit on the right-hand side of the equation, the number of nuclei present is N, the upper limit on the left-hand side of the equation. Upon integrating we get

$$\ln N \,|_{N_0}^{N} = -\lambda t \,|_0^t$$
$$\ln N - \ln N_0 = -\lambda t$$
$$\ln \frac{N}{N_0} = -\lambda t$$

Since $e^{\ln x} = x$, we now take e to both sides of the equation to get

$$\frac{N}{N_0} = e^{-\lambda t}$$

and solving for N we obtain

$$N = N_0 e^{-\lambda t} \tag{5.8}$$

Equation 5.8 is the radioactive decay law and it gives the total number of nuclei N present at any instant of time t. N_0 is the number of nuclei present at the time t = 0, which is the time that the observations of the nuclei is started. A plot of the radioactive decay law, equation 5.8, is shown in figure 5.3. The curve represents the number of radioactive nuclei still present at any time *t*. A very interesting quantity is found by looking for the time it takes for half of the original nuclei to decay,

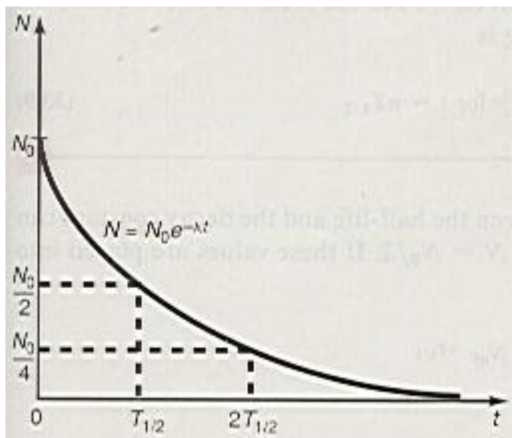

Figure 5.3 The radioactive decay law.

so that only half of the original nuclei are still present. Half the original nuclei is $N_0/2$ and is shown in the figure. A horizontal line for the value of $N_0/2$ is drawn until it intersects the curve $N = N_0e^{-\lambda t}$. A vertical line is dropped from this point to the *t*-axis. The value of the time read on the *t*-axis is the time it takes for half the original nuclei to decay. Hence, this time read from the *t*-axis is called the half-life of the radioactive nuclei and is denoted by $T_{1/2}$. *The **half-life** of a radioactive substance is thus the time it takes for half the original radioactive nuclei to decay.*

Example 5.2

The number of radioactive nuclei for several half-lives. One mole of a radioactive substance starts to decay. How many radioactive nuclei will be left after t = (a) $T_{1/2}$, (b) $2T_{1/2}$, (c) 3 $T_{1/2}$, (d) 4 $T_{1/2}$, and (e) $nT_{1/2}$ half-lives?

Solution

Since one mole of any substance contains 6.022×10^{23} atoms/mole (Avogadro's number), $N_0 = 6.022 \times 10^{23}$ nuclei.

a. At the end of one half-life, there will be

$$\frac{N_0}{2} = \frac{6.022 \times 10^{23} \text{ nuclei}}{2} = 3.011 \times 10^{23} \text{ nuclei}$$

b. At the end of another half-life, $t = 2T_{1/2}$, half of those present at the time $t = T_{1/2}$ will be lost, or

$$N = \frac{1}{2}\frac{N_0}{2} = \frac{N_0}{4} = \frac{6.022 \times 10^{23} \text{ nuclei}}{4}$$
$$= 1.506 \times 10^{23} \text{ nuclei}$$

c. At $t = 3T_{1/2}$, the number of radioactive nuclei remaining is

$$N = \frac{1}{2}\frac{N_0}{4} = \frac{N_0}{8} = 0.753 \times 10^{23} \text{ nuclei}$$

d. At $t = 4T_{1/2}$, the number of radioactive nuclei remaining is

$$N = \frac{1}{2}\frac{N_0}{8} = \frac{N_0}{16} = 0.376 \times 10^{23} \text{ nuclei}$$

e. At a period of time equal to n half-lives, we can see from the above examples that the number of nuclei remaining is

$$N = \frac{N_0}{2^n} \qquad \text{for } t = nT_{1/2} \tag{5.9}$$

Go to Interactive Example

An important relationship between the half-life and the decay constant can be found by noting that when $t = T_{1/2}$, $N = N_0/2$. If these values are placed into the decay law in equation 5.8, we get

$$\frac{N_o}{2} = N_o e^{-\lambda T_{1/2}}$$

or

$$\frac{1}{2} = e^{-\lambda T_{1/2}}$$

Taking the natural logarithm of both sides of this equation, we get

$$\ln\frac{1}{2} = \ln e^{-\lambda T_{1/2}} \tag{5.10}$$

But the natural logarithm ln is the inverse of the exponential function *e,* and when applied successively, as in the right-hand side of equation 5.10, they cancel each other leaving only the function. Hence, equation 5.10 becomes

$$\ln\frac{1}{2} = -\lambda T_{1/2}$$

Taking the natural logarithm ln of ½ on the electronic calculator gives –0.693. Thus,

$$-0.693 = -\lambda T_{1/2}$$

Solving for the decay constant, we get

$$\lambda = \frac{0.693}{T_{1/2}} \tag{5.11}$$

Thus, if we know the half-life $T_{1/2}$ of a radioactive nuclide, we can find its decay constant λ from equation 5.11. Conversely, if we know λ, then we can find the half-life from equation 5.11.

Example 5.3

Finding the decay constant and the activity for ${}^{90}_{38}Sr$ The half-life of strontium-90, ${}^{90}_{38}\mathrm{Sr}$, is 28.8 yr. Find (a) its decay constant and (b) its activity for 1 g of the material.

Solution

a. The decay constant, found from equation 5.11, is

$$\lambda = \frac{0.693}{T_{1/2}}$$
$$= \left(\frac{0.693}{28.8\text{ yr}}\right)\left(\frac{1\text{ yr}}{365\text{ days}}\right)\left(\frac{1\text{ day}}{24\text{ hr}}\right)\left(\frac{1\text{ hr}}{3600\text{ s}}\right)$$
$$= 7.63 \times 10^{-10}\text{ /s}$$

b. Before the activity can be determined, the number of nuclei present must be known. The atomic mass of ${}^{90}_{38}\mathrm{Sr}$ is 89.907746. Thus, 1 mole of it has a mass of approximately 89.91 g. The mass of 1 mole contains Avogadro's number or 6.022 × 10^{23} molecules. We find the number of molecules in 1 g of the material from the ratio

$$\frac{N_0}{N_A} = \frac{1\text{ g}}{89.91\text{ g}}$$

or the number of nuclei in 1 g of strontium-90 is

$$N_0 = \frac{1\text{ g}}{89.91\text{ g}}(N_A)$$

$$= \frac{1}{89.91}(6.022 \times 10^{23}) = 6.70 \times 10^{21} \text{ nuclei}$$

We can now find the activity from equation 5.7 as

$$A_0 = \lambda N_0$$
$$= (7.63 \times 10^{-10} \text{ 1/s})(6.70 \times 10^{21} \text{ nuclei})$$
$$= 5.11 \times 10^{12} \text{ nuclear disintegrations/s}$$

Go to Interactive Example

Note that the activity (that is, the number of disintegrations per second) is not a constant because it depends on N, which is decreasing with time by equation 5.8. In fact, if equation 5.8 is substituted into equation 5.7 for the activity, we get

$$A = \lambda N = \lambda N_0 e^{-\lambda t}$$

Letting

$$\lambda N_0 = A_0$$

the rate at which the nuclei are decaying at the time $t = 0$, we obtain for the activity

$$A = A_0 e^{-\lambda t} \tag{5.12}$$

Recalling that the activity is the number of disintegrations per second, we see that the rate of decay is not a constant but decreases exponentially. A plot of the activity as a function of time is shown in figure 5.4(a). Notice the similarity of this diagram with figure 5.3. For the time t, equal to a half-life, the activity is

$$A = A_0 e^{-\lambda T_{1/2}}$$

Substituting λ from equation 5.11, gives

$$A = A_0 e^{-[0.693/T_{1/2}]T_{1/2}} = A_0 e^{-0.693}$$

Using the electronic calculator, we obtain $e^{-0.693} = 0.500 = ½$. Hence,

$$A = \frac{A_0}{2} \qquad \text{for } t = T_{1/2} \tag{5.13}$$

That is, the rate of decay is cut in half for a time period of one half-life.

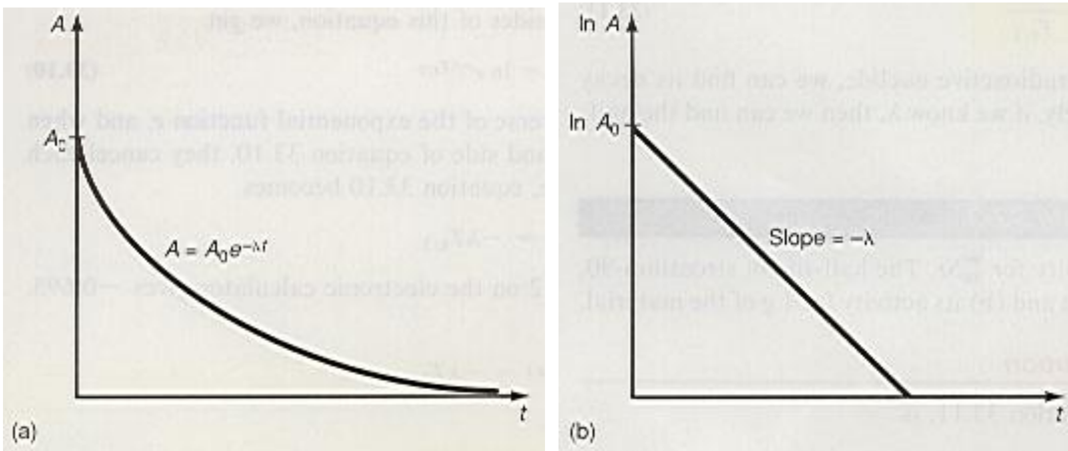

Figure 5.4 Radioactive activity.

Example 5.4

The number of radioactive nuclei and their rate of decay. Find the number of radioactive nuclei and their rate of decay for $t = T_{1/2}$ in the 1.00-g sample in example 5.3.

Solution

The number of nuclei left at the end of one half-life, found from equation 5.9, is

$$N = \frac{N_0}{2}$$

$$= \frac{6.70 \times 10^{21}\text{ nuclei}}{2}$$

$$= 3.35 \times 10^{21}\text{ nuclei}$$

While the rate of decay at the end of one half-life is found from equation 5.13 as

$$A = \frac{A_0}{2}$$

$$= \frac{5.11 \times 10^{12}\text{ decays/s}}{2}$$

$$= 2.55 \times 10^{12}\text{ decays/s}$$

Thus, at the end of 28 years, the number of strontium-90 radioactive nuclei have been cut in half and the rate at which they decay is also cut in half. That is, *there are less radioactive nuclei present at the end of the half-life, but the rate at which they decay also decreases*.

Go to Interactive Example

Example 5.5

The decay constant and activity for $^{91}_{38}\mathrm{Sr}$ The half-life of $^{91}_{38}\mathrm{Sr}$ is 9.70 hr. Find (a) its decay constant and (b) its activity for 1.00 g of the material.

Solution

a. This problem is very similar to example 5.3 except that this isotope of strontium has a very short half-life. The decay constant, found from equation 5.11, is

$$\lambda = \frac{0.693}{T_{1/2}}$$
$$= \left(\frac{0.693}{9.70\ \mathrm{hr}}\right)\left(\frac{1\ \mathrm{hr}}{3600\ \mathrm{s}}\right)$$
$$= 1.99 \times 10^{-5}\ /\mathrm{s}$$

b. The number of nuclei in a 1-g sample is found as before but the atomic mass of $^{91}_{38}\mathrm{Sr}$ is 90.90. Hence,

$$N_0 = \frac{1\ \mathrm{g}}{90.90\ \mathrm{g}}(N_A)$$
$$= \frac{1}{90.90}(6.022 \times 10^{23}\ \text{nuclei})$$
$$= 6.63 \times 10^{21}\ \text{nuclei}$$

The activity, again found from equation 5.7, is

$$A_0 = \lambda N_0$$
$$= (1.99 \times 10^{-5}\ /\mathrm{s})(6.63 \times 10^{21}\ \text{nuclei})$$
$$= 1.32 \times 10^{17}\ \text{disintegrations/s}$$

Comparing this example to example 5.3, we see *that for a smaller half-life, we get a larger decay constant λ, and hence a greater activity, or decays per second*.

Go to Interactive Example

The decay constant λ can be found experimentally using the following technique. First, let us return to equation 5.12 and take the natural logarithm of both sides of the equation, that is,

$$\ln A = \ln(A_0 e^{-\lambda t})$$

Now from the rules of manipulating logarithms, the logarithms of a product is equal to the sum of the logarithms of each term. Therefore,

$$\ln A = \ln A_0 + \ln e^{-\lambda t}$$

But as mentioned before, the natural log and the exponential are inverses of each other, and hence

$$\ln e^{-\lambda t} = -\lambda t$$

Thus,

$$\ln A = \ln A_0 - \lambda t$$

Rearranging, this becomes

$$\ln A = -\lambda t + \ln A_0 \tag{5.14}$$

If we now go into the laboratory and count the number of disintegrations per unit time A, at different times t, we can plot the ln A on the y-axis versus t on the x-axis, and obtain the straight line shown in figure 5.4(b). The slope of the line is $-\lambda$, and thus, λ can be determined experimentally. Once we know λ, we determine the half-life from equation 5.11.

We should also note that sometimes it is convenient to use the mean life or average life T_{avg} of a sample. The mean or average life is defined as the average lifetime of all the particles in a given sample of the material. It turns out to be just the reciprocal of the decay constant, that is,

$$T_{avg} = \frac{1}{\lambda} \tag{5.15}$$

5.4 Forms of Radioactivity

Up to now, only the number of decaying nuclei has been discussed, without specifying the details of the disintegrations. Nuclei can decay by:

1. Alpha decay, α
2. Beta decay, β^-
3. Beta decay, β^+, positron emission
4. Electron capture
5. Gamma decay, γ

Now let us discuss each of these in more detail.

Alpha Decay

When a nucleus has too many protons compared to the number of neutrons, the electrostatic force of repulsion starts to dominate the nuclear force of attraction. When this occurs, the nucleus is unstable and emits an α particle in radioactive decay. The nucleus thus loses two protons and two neutrons. Hence, its atomic number *Z,* which represents the number of protons in the nucleus, decreases by 2, while its mass number *A,* which is equal to the number of protons and neutrons in the nucleus, decreases by 4. Before the decay, the nucleus is called the "*parent" nucleus;* after the decay the nucleus is referred to as the *"daughter" nucleus.* Hence, *we represent an **alpha decay** symbolically as*

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}X + {}^{4}_{2}\mathrm{He} \tag{5.16}$$

where ${}^{A}_{Z}X$ is the parent nucleus, which decays into the daughter nucleus ${}^{A-4}_{Z-2}X$, and ${}^{4}_{2}\mathrm{He}$ is the α particle, which is the helium nucleus. *Notice that the atomic number Z has decreased by two units. This means that in alpha decay, one chemical element of atomic number Z has been transmuted into a new chemical element of atomic number Z− 2.* The dream of the ancient alchemists was to transmute the chemical elements, in particular to turn the baser metals into gold. This result was never attained because they were working with chemical reactions, which as has been seen, depends on the electronic structure of the atom and not its nucleus.

An example of a naturally occurring alpha decay can be found in uranium-238, which decays by α particle emission with a half-life of 4.51×10^9 yr. We find its daughter nucleus by using equation 5.16. Hence,

$$ {}^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}X + {}^{4}_{2}\mathrm{He} \tag{5.17}$$

Notice that the atomic number Z has dropped from 92 to 90. Consulting the table of the elements, we see that the chemical element with Z = 90 is thorium. Hence, uranium has been transmuted to thorium by the emission of an α particle. Equation 5.17 is now written as

$$ {}^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He} \tag{5.18}$$

Also note from the periodic table that the mass number A for thorium is 232, whereas in equation 5.18, the mass number is 234. This means that an isotope of thorium has been formed. (As a matter of fact ${}^{234}_{90}\mathrm{Th}$ is an unstable isotope and it also decays, only this time by beta emission. We will say more about this later.)

Beta Decay, β⁻

In beta decay, an electron is observed to leave the nucleus. However, as seen in chapter 3, an electron cannot be contained in a nucleus because of the Heisenberg uncertainty principle. Hence, the electron must be created within the nucleus at the moment of its emission. In fact, it has been found that a neutron within the nucleus decays into a proton and an electron, plus another particle called an *antineutrino.* The antineutrino is designated by the Greek letter nu, ν, with a bar over the ν, that is $\bar{\nu}$ The antineutrino is the antiparticle of the neutrino ν. The neutron decay is written as

$$ {}^{1}_{0}\mathrm{n} \rightarrow {}^{1}_{1}\mathrm{p} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} \tag{5.19}$$

The notation ${}^{0}_{-1}$ is used to designate the electron or β^- particle. It has a mass number A of 0 because it has no nucleons, and an atomic number of -1 to signify that it is a negative particle. The proton is written as ${}^{1}_{1}\mathrm{p}$ because it has a mass number and atomic number of one. Hence, in beta decay the nucleus loses a neutron but gains a proton, while the β^- particle, the electron, and the antineutrino are emitted from the nucleus. Thus, the atomic number Z increases by 1 in the decay because the nucleus gained a proton, but the mass number A stays the same because even though 1 neutron is lost, we have gained 1 proton.

It is perhaps appropriate to mention an interesting historical point here. The original assumption about neutron decay shown in equation 5.19 did not contain the antineutrino particle. The original decay seemed to violate the principle of conservation of energy. However, Wolfgang Pauli proposed the existence of a particle to account for the missing energy. Since the particle had to be neutral because of the law of conservation of electrical charge, the new particle was called a "neutrino" by the Italian-American physicist Enrico Fermi (1902-1954), for the "little" neutral particle. The antineutrino is the antiparticle of the neutrino, it is a particle of the same mass (zero rest mass) but has a spin component opposite to that of the neutrino. The neutrino was found experimentally in 1956. It is such an elusive particle that some move right through the earth without ever hitting anything.

A **beta decay,** β^- can be written symbolically as

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}X + {}^{0}_{-1}\mathrm{e} + \bar{\nu} \tag{5.20} $$

Note that in beta decay, Z increases to $Z + 1$. *Hence, a chemical element of atomic number Z is transmuted into another chemical element of atomic number Z + 1.*

As an example, the isotope ${}^{234}_{90}\mathrm{Th}$ is unstable and decays by beta emission with a half-life of 24 days. Its decay can be represented with the use of equation 5.20 as

$$ {}^{234}_{90}\mathrm{Th} \rightarrow {}^{234}_{91}X + {}^{0}_{-1}\mathrm{e} + \bar{\nu} $$

Looking up the periodic table of the elements, we see that the chemical element corresponding to the atomic number 91 is protactinium (Pa). Hence, the element thorium has been transmuted to the element protactinium. Also note from the periodic table that the mass number A for protactinium should be 231. Since we have a mass number of 234, this is an isotope of protactinium. (One that is also unstable and decays again.) The beta decay of thorium is now written as

$$ {}^{234}_{90}\mathrm{Th} \rightarrow {}^{234}_{91}\mathrm{Pa} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} $$

Example 5.6

Beta decay, β^-. The element ${}^{234}_{91}\mathrm{Pa}$ is unstable and decays by beta emission with a half-life of 6.66 hr. Find the nuclear reaction and the daughter nuclei.

Solution

Because ${}^{234}_{91}\mathrm{Pa}$ decays by beta emission, it follows the form of equation 5.20. Hence,

$${}^{234}_{91}\mathrm{Pa} \rightarrow {}^{234}_{92}X + {}^{0}_{-1}\mathrm{e} + \bar{\nu}$$

But from the table of elements, $Z = 92$ is the atomic number of uranium. Hence, the daughter nuclei is ${}^{234}_{92}\mathrm{U}$, and the entire reaction is written as

$${}^{234}_{91}\mathrm{Pa} \rightarrow {}^{234}_{92}\mathrm{U} + {}^{0}_{-1}\mathrm{e} + \bar{\nu}$$

Beta Decay, β^+ Positron Emission

In this type of decay, a positron is emitted from the nucleus. A *positron* is the antiparticle of the electron. It has all the characteristics of the electron except it carries a positive charge. Because there are no positrons in the nucleus, a positron must be created immediately before emission. Positron emission is the result of the decay of a proton into a neutron, a positron, and a neutrino ν, and is written symbolically as

$${}^{1}_{1}\mathrm{p} \rightarrow {}^{1}_{0}\mathrm{n} + {}^{0}_{+1}\mathrm{e} + \nu \tag{5.21}$$

The positron ${}^{0}_{+1}\mathrm{e}$ is emitted with the neutrino ν. The neutron stays behind in the nucleus. *Hence, in a* ***beta decay,*** $\boldsymbol{\beta}^+$ *the atomic number Z decreases by one because of the loss of the proton. The mass number A stays the same because even though a proton is lost, a neutron is created to keep the same number of nucleons. Hence, a* β^+ *decay can be written symbolically as*

$${}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}X + {}^{0}_{+1}\mathrm{e} + \nu \tag{5.22}$$

As an example, the isotope of aluminum ${}^{26}_{13}\mathrm{Al}$ is unstable and decays by β^+ emission with a half-life of 7.40×10^5 yr. The reaction is written with the help of equation 5.22 as

$${}^{26}_{13}\mathrm{Al} \rightarrow {}^{26}_{12}X + {}^{0}_{+1}\mathrm{e} + \nu$$

Looking at the periodic table of the elements, we find that the atomic number 12 corresponds to the chemical element magnesium Mg. Hence,

$${}^{26}_{13}\mathrm{Al} \rightarrow {}^{26}_{12}\mathrm{Mg} + {}^{0}_{+1}\mathrm{e} + \nu$$

Because the mass number A of magnesium is 24, we see that this transmutation created an isotope of magnesium.

It is important to note that the decay of the proton, equation 5.21, can only occur within the nucleus. A free proton cannot decay into a neutron because the mass of the proton is less than the mass of the neutron.

Electron Capture

Occasionally an orbital electron gets too close to the nucleus and gets absorbed by the nucleus. Since the electron cannot remain as an electron within the nucleus, it combines with a proton and in the process creates a neutron and a neutrino. We represent this as

$$ {}^{0}_{-1}\mathrm{e} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{1}_{0}\mathrm{n} + \nu \tag{5.23}$$

The net result of this process decreases the number of protons in the nucleus by one hence changing Z to $Z - 1$, while keeping the number of nucleons A constant. Hence, *this decay can be written as*

$$ {}^{0}_{-1}\mathrm{e} + {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}X + \nu \tag{5.24}$$

When the electron that is close to the nucleus is captured by the nucleus, it leaves a vacancy in the electron orbit. An electron from a higher energy orbit falls into this vacancy. The difference in the energy of the electron in the higher orbit from the energy in the lower orbit is emitted as a photon in the X-ray portion of the spectrum.

As an example of electron capture, we consider the isotope of mercury ${}^{197}_{80}\mathrm{Hg}$ that decays by electron capture with a half-life of 65 hr. This decay can be represented, with the help of equation 5.24, as

$$ {}^{0}_{-1}\mathrm{e} + {}^{197}_{80}\mathrm{Hg} \rightarrow {}^{197}_{79}X + \nu $$

Consulting the table of elements, we find that the atomic number $Z = 79$ represents the chemical element gold, Au. Hence,

$$ {}^{0}_{-1}\mathrm{e} + {}^{197}_{80}\mathrm{Hg} \rightarrow {}^{197}_{79}\mathrm{Au} + \nu $$

Thus, the dreams of the ancient alchemists have been fulfilled. An isotope of mercury has been transmuted into the element gold. Also note that mass number A of gold is 197. Hence, the transmutation has given the stable element gold.

Gamma Decay

A nucleus undergoing a decay is sometimes left in an excited state. Just as an electron in an excited state of an atom emits a photon and drops down to the ground state, a proton or neutron can be in an excited state in the nucleus. When the nucleon drops back to its ground state, it also emits a photon. Because the energy given off is so large, the frequency of the photon is in the gamma ray portion of the electromagnetic spectrum. Hence, the excited nucleus returns to its ground state and a gamma ray is emitted. Thus *gamma decay is represented symbolically as*

$$ {}^{A}_{Z}X^{*} \rightarrow {}^{A}_{Z}X + \gamma \tag{5.25}$$

Where the * on the nucleus indicates an excited state. *In this type of decay, neither the atomic number Z nor the mass number A changes. Hence, gamma decay does not transmute any of the chemical elements.*

5.5 Radioactive Series

As indicated earlier, elements with atomic numbers Z greater than 83 are unstable and decay naturally. Most of these unstable elements have very short lifetimes and decay rather quickly. Hence, they are not easily found in nature. The exceptions to this are the elements thorium-232, uranium-238, and the uranium isotope 235. The element ${}^{232}_{90}\mathrm{Th}$ has a half-life of 1.39×10^{10} yr, ${}^{238}_{92}\mathrm{U}$ has a half-life of 4.50×10^{9} yr, and ${}^{235}_{92}\mathrm{U}$ has a half-life of 7.10×10^{8} yr. Moreover, these elements decay into a series of daughters, granddaughters, great granddaughters, and so on.

As an example, the series decay ${}^{232}_{90}\mathrm{Th}$ is shown in figure 5.5, which is a plot of the neutron number N versus the atomic number Z. Because ${}^{232}_{90}\mathrm{Th}$ has a Z value of 90 and an N value of $232 - 90 = 142$, ${}^{232}_{90}\mathrm{Th}$ is plotted with the coordinates $N = 142$ and $Z = 90$. First ${}^{232}_{90}\mathrm{Th}$ decays by alpha emission with a half-life of 1.39×10^{10} yr. As seen in section 5.4, equation 5.16, an alpha decay changes the atomic number Z to $Z - 2$, and decreases the mass number A by 4 to $A - 4$. Thus, ${}^{232}_{90}\mathrm{Th}$ decays as

$${}^{232}_{90}\mathrm{Th} \rightarrow {}^{228}_{88}X + {}^{4}_{2}\mathrm{He}$$

But atomic number 88 corresponds to the chemical element radium (Ra). Hence,

$${}^{232}_{90}\mathrm{Th} \rightarrow {}^{228}_{88}\mathrm{Ra} + {}^{4}_{2}\mathrm{He}$$

The neutron number N for ${}^{228}_{88}\mathrm{Ra}$ is $228 - 88 = 140$. Thus, ${}^{228}_{88}\mathrm{Ra}$ is found in the diagram with coordinates, $N = 140$ and $Z = 88$. The original neutron number is given by

$$N_0 = A - Z$$

But in alpha emission, A goes to $A - 4$ and Z goes to $Z - 2$, equation 5.16. Hence, the new neutron number is given by

$$N_1 = (A - 4) - (Z - 2)$$

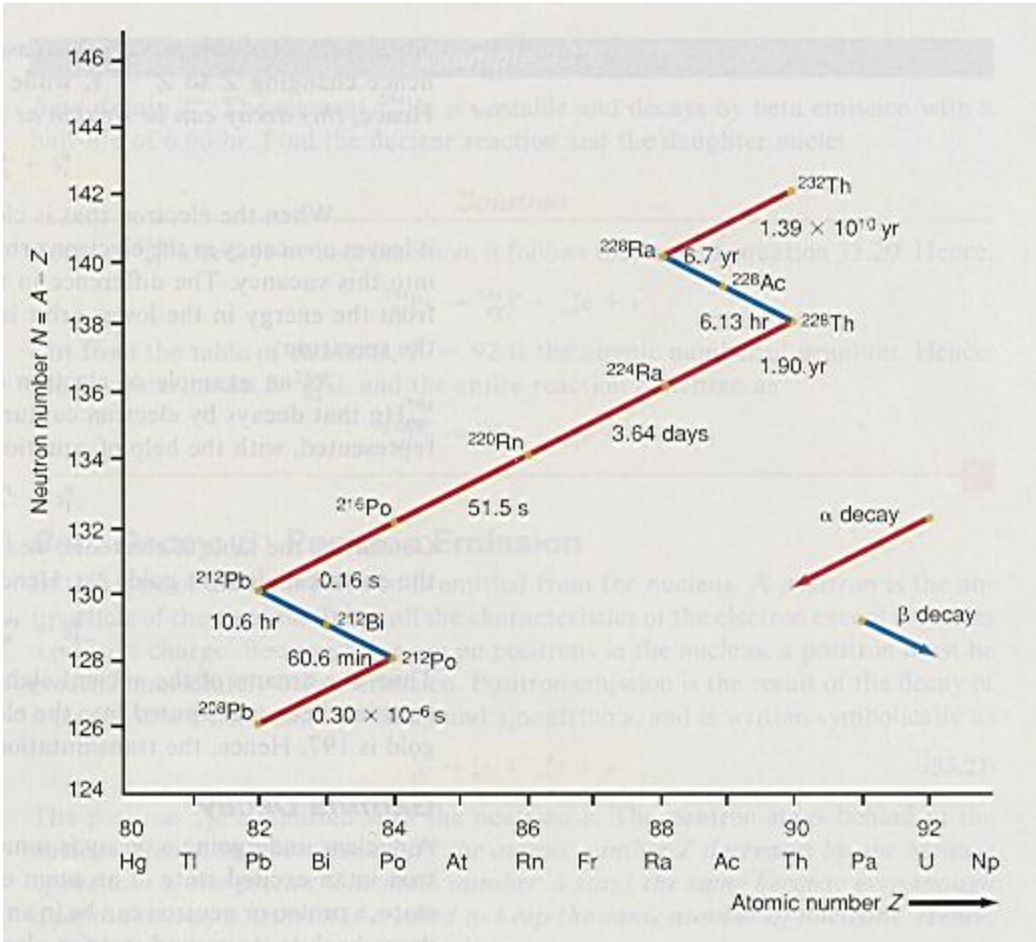

Figure 5.5 Thorium $^{232}_{90}\text{Th}$ decay series.

$$= A - Z - 2$$
$$N_1 = N_0 - 2 \qquad \text{(alpha decay)} \tag{5.26}$$

Thus, for all alpha emissions, the neutron number decreases by 2. Hence, in the diagram, for every alpha emission the element has both N and Z decreased by 2.

Radium-228 is also unstable and decays by beta emission with a half-life of 6.7 yr. As shown in equation 5.20, the value of the atomic number Z increases to Z + 1, while the mass number A remains the same. The neutron number for beta emission becomes

$$N_1 = A - (Z + 1) = A - Z - 1$$
$$N_1 = N_0 - 1 \qquad \text{(beta}^- \text{ decay)} \tag{5.27}$$

Thus, $^{228}_{88}\text{Ra}$ becomes actinium, $^{228}_{89}\text{Ac}$, with coordinates $N = 139$ and $Z = 89$.

Therefore, in the series diagram, alpha emission appears as a line sloping down toward the left, with both N and Z decreasing by 2 units. Beta emission, on the other hand, appears as a line sloping downward to the right with N decreasing by 1 *and Z increasing by 1*. The entire decay of the family is shown in figure 5.5: thorium-232 decays by α emission to radium-228, which then decays by β^- emission to actinium-228, which then decays by β^- to thorium-228, which then decays by α emission to radium-224, which then decays by α emission to radon-220, which then decays by α emission to polonium-216, which then decays by α emission to lead-212, which then decays by β^- to bismuth-212, which then decays by β^- to polonium-212, which finally decays by α emission to the stable lead-208. The half-life for each decay is shown in the diagram.

The radioactive chain is called a *series*. The decay series for uranium- 238 is shown in figure 5.6. It starts with ${}^{238}_{92}\mathrm{U}$ and ends in the stable isotope of lead-206. Figure 5.7 shows the decay series for uranium-235. As we can see, the series ends with the stable chemical element lead-207. Figure 5.8 shows the neptunium series that ends in the stable chemical element bismuth-209. Neptunium is called a *transuranic element* because it lies beyond uranium in the periodic table. Uranium with an atomic number Z = 92 is the highest chemical element found in nature. Elements with Z greater than 92 have been made by man. Many different isotopes of these new elements can also be created.

As an example of the creation of a transuranic element, bombarding ${}^{238}_{92}\mathrm{U}$ with neutrons creates neptunium by the reaction

$$ {}^{238}_{92}\mathrm{U} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{239}_{93}\mathrm{Np} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} \tag{5.28} $$

That is, ${}^{238}_{92}\mathrm{U}$ absorbs the neutron and then goes through a beta decay by emitting an electron. The atomic number is increased by one, from Z = 92 to Z = 93, thus creating an isotope of a new chemical element, which is called neptunium.

Neptunium-239 is itself unstable and decays by beta emission creating still another chemical element called plutonium, according to the reaction

$$ {}^{239}_{93}\mathrm{Np} \rightarrow {}^{239}_{94}\mathrm{Pu} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} \tag{5.29} $$

The next transuranic element to be created was americium, which was created by the series of processes given by

$$ \begin{gathered} {}^{239}_{94}\mathrm{Pu} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{240}_{94}\mathrm{Pu} + \gamma \\ {}^{240}_{94}\mathrm{Pu} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{241}_{94}\mathrm{Pu} + \gamma \\ {}^{241}_{94}\mathrm{Pu} \rightarrow {}^{241}_{95}\mathrm{Am} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} \end{gathered} $$

That is, plutonium is bombarded with neutrons until the isotope ${}^{241}_{94}\mathrm{Pu}$ is created, which then beta decays producing the isotope of the new chemical element americium. Bombarding elements with various other particles and elements, created still more elements. As examples of the creation of some other new elements, we have

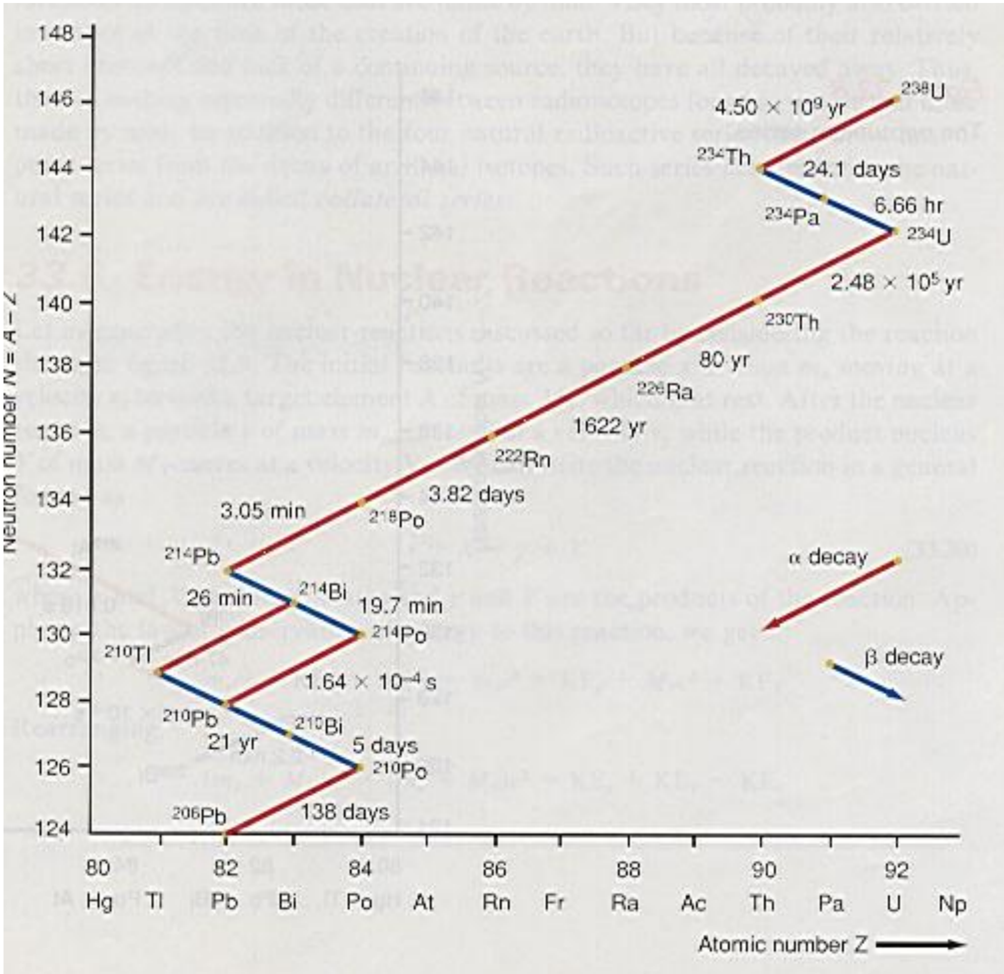

Figure 5.6 Uranium ${}^{238}_{92}\text{U}$ decay series.

(curium)	${}^{239}_{94}\text{Pu} + {}^{4}_{2}\text{He} \rightarrow {}^{242}_{96}\text{Cm} + {}^{1}_{0}\text{n}$
(berkelium)	${}^{241}_{94}\text{Am} + {}^{4}_{2}\text{He} \rightarrow {}^{243}_{97}\text{Bk} + {}^{0}_{-1}\text{e} + \bar{\nu} + 2\,{}^{1}_{0}\text{n}$
(californium)	${}^{242}_{96}\text{Cm} + {}^{4}_{2}\text{He} \rightarrow {}^{245}_{98}\text{Cf} + {}^{1}_{0}\text{n}$
(nobelium)	${}^{246}_{96}\text{Cm} + {}^{12}_{6}\text{C} \rightarrow {}^{254}_{102}\text{No} + 4\,{}^{1}_{0}\text{n}$
(lawrencium)	${}^{252}_{98}\text{Cf} + {}^{10}_{5}\text{B} \rightarrow {}^{257}_{103}\text{Lr} + 5\,{}^{1}_{0}\text{n}$

The neptunium decay series was later found to actually start with plutonium, ${}^{241}_{94}\text{Pu}$, which decays by beta emission to americium, ${}^{241}_{95}\text{Am}$, which then decays by alpha emission to neptunium, ${}^{237}_{93}\text{Np}$.

Because of the very long lifetimes of the parent element of these series, most of the members of the series are found naturally. An equilibrium condition is established within the series with as many isotopes decaying as are being formed. Artificial isotopes are those that are made by man. They most probably also existed

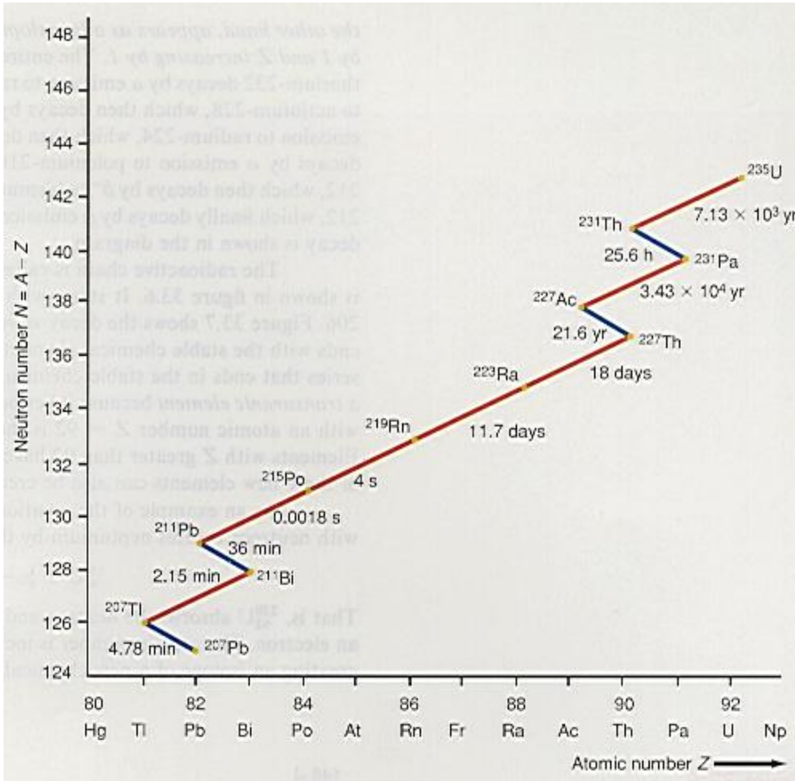

Figure 5.7 Uranium $^{235}_{92}U$ decay series.

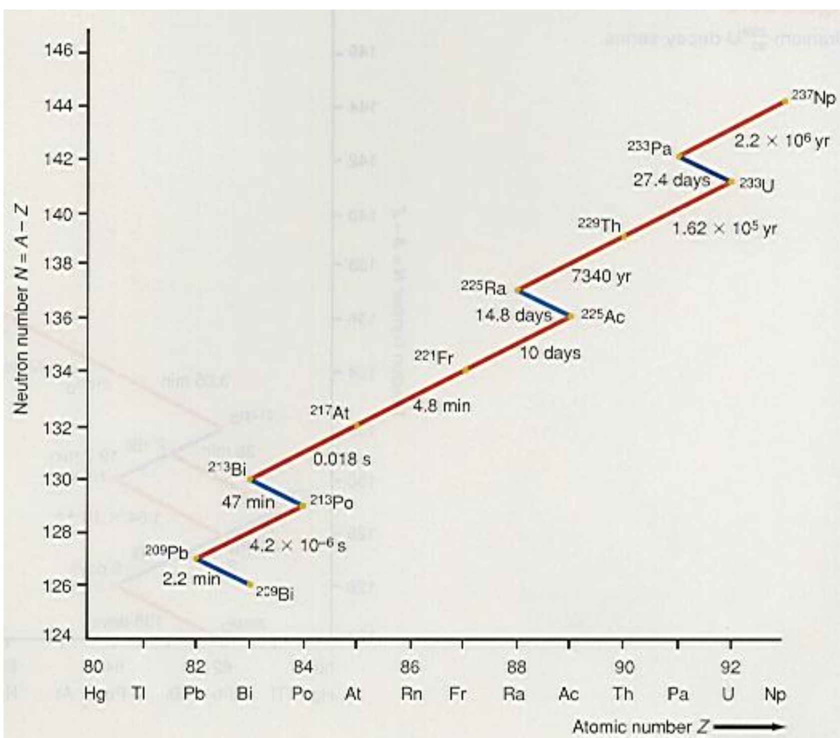

Figure 5.8 The neptunium series.

in nature at the time of the creation of the earth. But because of their relatively short lifetimes and lack of a continuing source, they have all decayed away. Thus, there is nothing essentially different between radioisotopes found in nature and those made by man. In addition to the four natural radioactive series there are a host of other series from the decay of artificial isotopes. Such series are similar to the natural series and are called *collateral series*.

5.6 Energy in Nuclear Reactions

Let us generalize the nuclear reactions discussed so far by considering the reaction shown in figure 5.9. The initial reactants are a particle x of mass m_x moving at a velocity $\mathbf{v}_x$ toward a target element X of mass M_X, which is at rest. After the nuclear reaction, a particle y of mass m_y leaves with a velocity $\mathbf{v}_y$ while the product nucleus

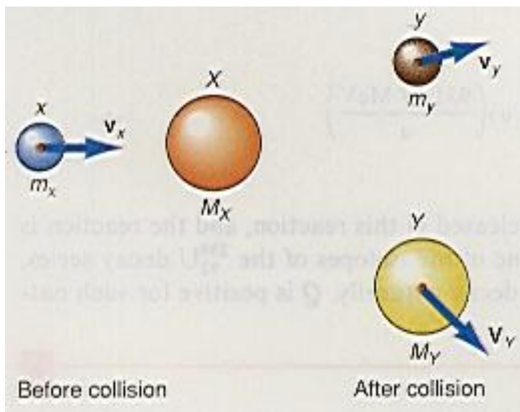

Figure 5.9 Nuclear reaction as a collision.

Y of mass M_Y moves at a velocity $\mathbf{V}_Y$. We can write the nuclear reaction in a general format as

$$x + X = y + Y \tag{5.30}$$

where x and X are the reactants and y and Y are the products of the reaction. Applying the law of conservation of energy to this reaction, we get

$$m_xc^2 + \mathrm{KE}_x + M_Xc^2 = m_yc^2 + \mathrm{KE}_y + M_Yc^2 + \mathrm{KE}_Y$$

Rearranging,

$$(m_x + M_X)c^2 - (m_y + M_Y)c^2 = \mathrm{KE}_y + \mathrm{KE}_Y - \mathrm{KE}_x$$

The ***Q* value of a nuclear reaction** *is now defined as the energy available in a reaction caused by the difference in mass between the reactants and the products.* Thus,

$$Q = (m_x + M_X)c^2 - (m_y + M_Y)c^2 \tag{5.31}$$

or

$$Q = [(\text{Input mass}) - (\text{Output mass})]c^2 \tag{5.32}$$

or

$$Q = E_{in} - E_{out} \tag{5.33}$$

That is, the Q value is the difference between the energy put into a nuclear reaction E_{in} and the energy that comes out E_{out}.

If $m_x + M_X$ is greater than $m_y + M_Y$, then Q is greater than zero ($Q > 0$). That is, the input mass energy is greater than the output mass energy. Thus, mass is lost in the nuclear reaction and an amount of energy Q is released in the process. *A nuclear reaction in which energy is released is called an* ***exoergic reaction*** *(sometimes called an exothermic reaction).*

Example 5.7

Energy released in a nuclear reaction. How much energy is released or absorbed in the following reaction?

$${}^{219}_{86}\text{Rn} \rightarrow {}^{215}_{84}\text{Po} + {}^{4}_{2}\text{He}$$

Solution

The mass of radon-219 is 219.009523 unified mass units (u) while the mass of polonium, ${}^{215}_{84}\text{Po}$, is 214.999469 u. The mass of the α particle is 4.002603 u. The total output mass is

$$\begin{array}{r} 214.999469\ \text{u} \\ \underline{+4.002603\ \text{u}} \\ 219.002072\ \text{u} \end{array}$$

Hence, the difference in mass between the input mass and the output mass is

$$219.009523\ \text{u} - 219.002072\ \text{u} = 0.007451\ \text{u}$$

Converting this to an energy

$$\begin{aligned} Q &= (0.007451\ \text{u})\left(\frac{931.49\ \text{MeV}}{\text{u}}\right) \\ &= 6.94\ \text{MeV} \end{aligned}$$

Since Q is greater than zero, energy is released in this reaction, and the reaction is exoergic. We might note that ${}^{219}_{86}\text{Rn}$ is one of the isotopes of the ${}^{235}_{92}\text{U}$ decay series. Because all of the isotopes of this chain decay naturally, Q is positive for such natural decays.

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If in a nuclear reaction, $m_x + M_X$ is less than $m_y + M_Y$, then the Q value is negative ($Q < 0$). In such a reaction, mass is created if an amount of energy Q is added to the system. This energy is usually added by way of the kinetic energy of the reacting particle and nuclei. *A nuclear reaction in which energy is added to the system is called an* ***endoergic reaction*** *(sometimes called an endothermic reaction).*

A nuclear reaction proceeds naturally in the direction of minimum energy. Thus, in the decay of a natural radioactive nuclide, the nucleus emits a particle in order to reach a lower equilibrium energy state. The excess energy is given off in the process. Endoergic reactions, on the other hand, do not occur naturally in the physical world because the energy of the reactants is less than the required energy for the products to be created. Thus, endoergic reactions cannot take place unless energy is added to the system. The energy is added by accelerating the particle to very high speeds in an accelerator. When the particle hits the target, this additional kinetic energy is the energy necessary to make the reaction proceed. It is sometimes necessary to have additional kinetic energy to overcome the Coulomb barrier.

Example 5.8

Find the Q value of a nuclear reaction. The first artificial transmutation of an element was performed by Rutherford in 1919 when he bombarded nitrogen with alpha particles according to the reaction

$$^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{p}$$

Find the Q value associated with this reaction.

Solution

The Q value, found from equation 5.31, is

$$Q = (m_x + M_X)c^2 - (m_y + M_Y)c^2$$

where

$$m_x = m({}^{4}_{2}\mathrm{H}) = 4.002603 \text{ u}$$
$$\underline{M_X = m\,({}^{14}_{7}\mathrm{N}) = 14.003242 \text{ u}}$$
$$(m_x + M_X) = 18.005845 \text{ u}$$

and

$$m_y = m_p = 1.007825 \text{ u}$$
$$\underline{M_Y = m({}^{17}_{8}\mathrm{O}) = 16.999133 \text{ u}}$$
$$(m_y + M_Y) = 18.006958 \text{ u}$$

Hence,

$$Q = 18.005845 \text{ u} - 18.006958 \text{ u}$$

$$= -(0.001113 \text{ u})\left(\frac{931.49 \text{ MeV}}{1 \text{ u}}\right)$$
$$= -1.04 \text{ MeV}$$

Therefore, Q is negative, and this much energy must be supplied to start the reaction. The initial α particle used by Rutherford had energies of about 5.5 MeV, well above the amount of energy needed.

Go to Interactive Example

We should note that the amount of energy necessary to break up the nucleus, its Q value, is the same as the binding energy of the nucleus, BE, discussed in section 5.2. We can now write a nuclear reaction in the form

$$x + X \rightarrow y + Y + Q \tag{5.34}$$

When $Q > 0$, Q is the amount of energy released in a reaction. When $Q < 0$, Q is the amount of energy that must be added to the system in order for the reaction to proceed.

5.7 Nuclear Fission

In 1934, Enrico Fermi, beginning at the bottom of the periodic table, fired neutrons at each chemical element in order to create isotopes of the elements. He systematically worked his way up the periodic table until he came to the last known element, at that time, uranium. He assumed that bombarding uranium with neutrons would make it unstable. He then felt that if the unstable uranium nucleus went through a beta decay, the atomic number would increase from 92 to 93 and he would have created a new element. (He was the first to coin the word *transuranic*.) However after the bombardment of uranium, he could not figure out what the products of the reaction were.

From 1935 through 1938, the experiments were repeated in Germany by Otto Hahn and Lise Meitner. The German chemist, Ida Noddack, analyzed the products of the reaction and said that *it appeared as if the uranium atom had been split into two lighter elements*. Lise Meitner and her nephew, Otto Frisch, considered these results and concluded that indeed the atom had been split into two lighter elements. The splitting of an atom resembled the splitting of one living cell into two cells of equal size. This biological process is called fission. Otto Frisch then used this biological term, fission, to describe the splitting of an atom. Hence, ***nuclear fission** is the process of splitting a heavy atom into two lighter atoms.* The isotope of uranium that undergoes fission is $^{235}_{92}U$. The process can be described in general as

$$ {}_0^1\text{n} + {}_{92}^{235}\text{U} \rightarrow y + Y + {}_0^1\text{n} + Q \tag{5.35} $$

The fission process does not always produce the same fragments, however. It was found that the product or fragment nuclei, y and Y, varied between the elements Z = 36 to Z = 60. Some typical fission reactions are:

$$ {}_0^1\text{n} + {}_{92}^{235}\text{U} \rightarrow {}_{56}^{141}\text{Ba} + {}_{36}^{92}\text{Kr} + 3\,{}_0^1\text{n} + Q \tag{5.36} $$

$$ {}_0^1\text{n} + {}_{92}^{235}\text{U} \rightarrow {}_{56}^{144}\text{Ba} + {}_{36}^{89}\text{Kr} + 3\,{}_0^1\text{n} + Q \tag{5.37} $$

$$ {}_0^1\text{n} + {}_{92}^{235}\text{U} \rightarrow {}_{54}^{140}\text{Xe} + {}_{38}^{94}\text{Sr} + 2\,{}_0^1\text{n} + Q \tag{5.38} $$

$$ {}_0^1\text{n} + {}_{92}^{235}\text{U} \rightarrow {}_{50}^{132}\text{Sn} + {}_{42}^{101}\text{Mo} + 3\,{}_0^1\text{n} + Q \tag{5.39} $$

In all cases, the masses of the product nuclei are less than the masses of the reactants, indicating that the Q value is greater than zero. The reaction is, therefore, exoergic and energy is given off in the process.

Example 5.9

The Q value of a nuclear fission reaction. Find the Q value associated with the nuclear fission process given by equation 5.36.

Solution

The mass of the reactants are

$$ \begin{aligned} m_\text{n} &= 1.008665\ \text{u} \\ \underline{m({}_{92}^{235}\text{U})} &\underline{= 235.043933\ \text{u}} \\ m_\text{n} + m({}_{92}^{235}\text{U}) &= 236.052598\ \text{u} \end{aligned} $$

The mass of the products are

$$ \begin{aligned} 3m_\text{n} &= 3.025995\ \text{u} \\ m({}_{56}^{141}\text{Ba}) &= 140.913740\ \text{u} \\ \underline{m({}_{36}^{92}\text{Kr})} &\underline{= 91.925765\ \text{u}} \\ m_\text{Ba} + m_\text{Kr} &= 235.865500\ \text{u} \end{aligned} $$

The mass lost in the process is

$$ \begin{aligned} \Delta m &= 236.052598\ \text{u} - 235.865500\ \text{u} \\ &= +\,0.187098\ \text{u} \end{aligned} $$

The Q value is obtained by multiplying Δm by the conversion factor 931.49 MeV = 1 u.

$$Q = (0.187098\text{ u})\left(\frac{931.49\text{ MeV}}{\text{u}}\right)$$
$$= 174\text{ MeV}$$

Hence, the splitting of only one nucleus of ${}^{235}_{92}U$ gives off an enormous quantity of energy. The actual energy in the fission process turns out to be even greater than this because the fragments themselves are radioactive and give off an additional 15 to 20 MeV of energy as they decay. Hence, in the entire fission process of ${}^{235}_{92}U$, some 200 MeV of energy are given off per nucleus.

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Example 5.10

The energy of fission of uranium. If 1 kg of ${}^{235}_{92}U$ were to go through the fission process, how much energy would be released?

Solution

Because the mass of any quantity is equal to the mass of one atom times the total number of atoms, that is,

$$m = m_{\text{atom}}N$$

the number of atoms is

$$N = \frac{m}{m_{\text{atom}}}$$
$$= \left(\frac{1\text{ kg}}{235.04\text{ u}}\right)\left(\frac{1\text{ u}}{1.66 \times 10^{-27}\text{kg}}\right)$$
$$= 2.56 \times 10^{24}\text{ atoms}$$

But the number of nuclei is exactly equal to the number of atoms, hence, there are 2.56×10^{24} uranium nuclei in 1 kg of uranium-235. Assuming a total energy release of 200 MeV per nucleus, the total energy released is

$$E = \left(\frac{200\text{ MeV}}{\text{nuclei}}\right)(2.56 \times 10^{24}\text{ nuclei})$$
$$= 5.12 \times 10^{26}\text{ MeV} = 8.19 \times 10^{13}\text{ J}$$

which is an absolutely immense amount of energy. It is comparable to the amount of energy released from the explosion of 20,000 tons of TNT.

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A theoretical model of nuclear fission, developed by Niels Bohr and John A. Wheeler, and called the *liquid-drop model,* is sketched in figure 5.10. When the

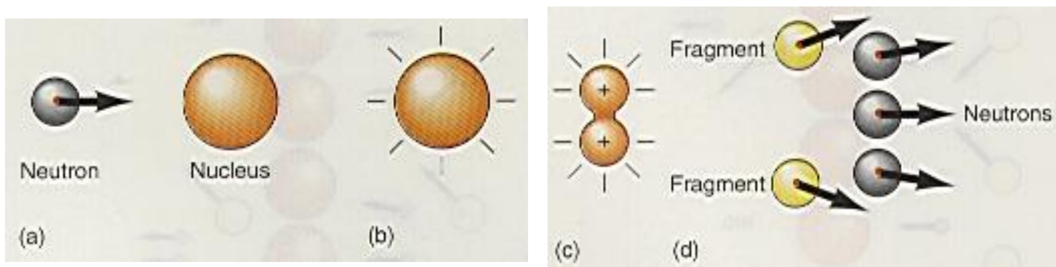


Figure 5.10 The liquid drop model of nuclear fission.

bombarding neutron is captured by the uranium nucleus, the nucleus becomes unstable, vibrates, and becomes deformed as in figure 5.10(c). In the deformed state, the nuclear force is not as great as usual because the nucleus is spread so far apart. The Coulomb force of repulsion is, however, just as strong as always and acts to split the drop (nucleus) into fragments, figure 5.10(d). Thus, the uranium nucleus is split into fragment nuclei accompanied by extra neutrons and a large amount of energy.

All in all, there are about 90 different daughter nuclei formed in the fission process. The initial neutrons that are used to bombard the uranium are called *slow neutrons* because they have very small kinetic energies and, hence, low velocities and, therefore, move slowly. The slow neutrons have a large probability of capture by the uranium-235 nucleus because they are moving so slowly. There are about two or three neutrons released per each fission.

A historical anecdote relating to nuclear fission might be interesting to mention here. In 1906, at McGill University in Montreal, Canada, Lord Rutherford said: “If it were ever found possible to control at will, the rate of disintegration of the radioactive elements, an enormous amount of energy could be obtained from a small quantity of matter.”[1] With age, Rutherford was to change that vision to, “The energy produced by breaking down the atom is a very poor kind of thing. Anyone who expects a source of power from the transformation of these atoms is talking moonshine.” That statement was a challenge to Leo Szilard (1898-1964), a Hungarian physicist working for Rutherford. Szilard thought, “What if you found an element in which nuclei throw off energy? What if you could make it happen at will? What if this element’s atoms threw off two new neutrons to strike two more nuclei. Two twos are four, four fours are sixteen — in a flash, the number would be astronomical. Moonshine? All you need do is to find the right element!”[2]

1[1] *Uranium, The Deadly Element,* by Lennard Bikel, p. 43.
2[2] Ibid, p. 72.

A by-product of fission is that it produces the same particles that initiated the fission in the first place, namely neutrons. If more neutrons are produced than started the reaction, the result is a multiplication. If the excess product neutrons can initiate more fission, more neutrons are produced to produce more fission, and so on and on. The result is a chain reaction, as shown in figure 5.11. The

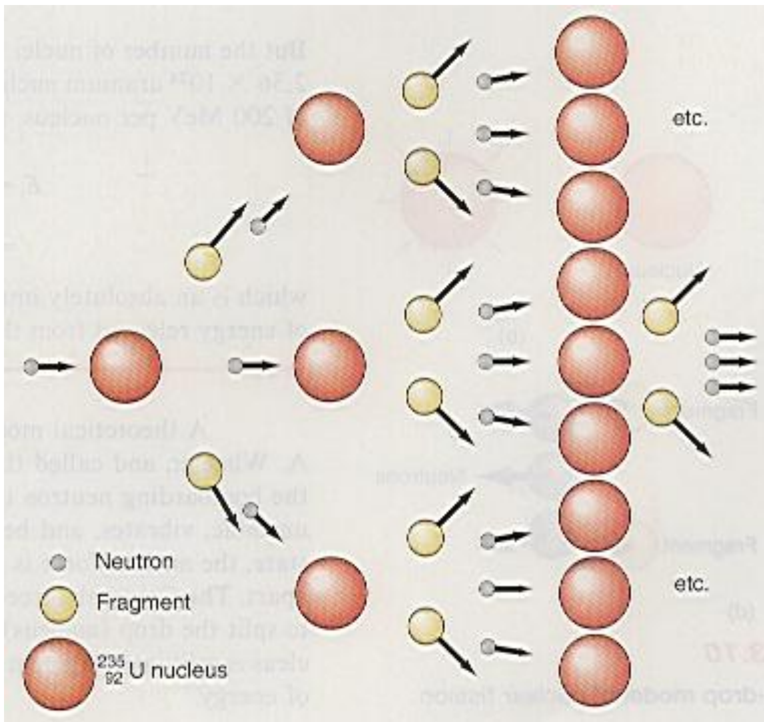


Figure 5.11 The chain reaction.

multiplication of neutrons is given by a multiplication factor, k. If $k < 1$, the reaction gives less neutrons than initiated the reaction, and the chain dies out. If $k > 1$ the reaction gives too many neutrons and the reaction escalates and runs wild. If $k = 1$, just the right number of neutrons are produced to keep the process going at a constant rate.

Natural uranium contains 99.3% of $^{238}_{92}U$ and only 0.7% of $^{235}_{92}U$, and cannot chain react. To get a chain reaction, the percentage of $^{235}_{92}U$ must be increased. Weapons grade uranium contains about 50% of $^{235}_{92}U$, whereas *nuclear reactor grade uranium contains only about 3.6% of* $^{235}_{92}U$, which is much too small to produce a nuclear explosion.

To finish our little story, Leo Szilard filed an application in the London Patents Office on June 28, 1934. It was the world's first registration of a nuclear process chain reaction by neutron bombardment. Szilard was afraid his chain reaction idea would fall into Nazi hands, so he assigned his patent to the British Admiralty. In general, the heavier the nucleus that is split, the greater the energy given off. Szilard made the mistake of proposing to split the lighter elements instead of the heavier ones.

The Atomic Bomb

The possibility of a chain reaction in uranium with its extremely large energy release, led some of the nuclear scientists to conceive of making a bomb — an atomic bomb. The Second World War was raging in Europe and the scientists were afraid that Hitler might develop such a bomb. Such a bomb in his hands, it was felt, would mean the end of the civilized world. For our own protection, it was imperative that we should develop such a bomb as quickly as possible. Leo Szilard and Edward Teller (later to become the Father of the hydrogen bomb), both Hungarian physicists who were refugees from Hitler's Europe, approached Albert Einstein and had him draft a letter to President Roosevelt on the possibility of making an atomic bomb. The letter was given to Dr. Alexander Sachs who personally delivered it to President Roosevelt on October 11, 1939. Ironically, the final decision to go ahead with the development of the A-bomb was made on December 6, 1941, under the name of the Manhattan Project.

In order to make an atomic bomb, enough uranium-235 had to be assembled to make the chain reaction. The amount of mass of uranium-235 needed to start the chain reaction was called the *critical mass*. The uranium-235 had to consist of two pieces, both below the critical mass. When one piece, in the form of a bullet, was fired into the second piece, the critical mass was obtained and the chain reaction would lead to a violent explosion. This was the type of bomb called the "Thin Man" that was detonated at Hiroshima on August 5, 1945. The difficulty with a uranium bomb was that it was relatively difficult to separate ${}^{235}_{92}\mathrm{U}$ from ${}^{238}_{92}\mathrm{U}$.

As already seen in equation 5.28, bombarding ${}^{238}_{92}\mathrm{U}$ with neutrons produces the element neptunium, ${}^{239}_{93}\mathrm{Np}$, which decays into plutonium, ${}^{239}_{94}\mathrm{Pu}$, equation 5.29. It turns out that plutonium-239 is even more fissionable than uranium-235, so a much smaller mass of it is necessary for its critical mass. By making a nuclear reactor, which we will discuss in a moment, a very large, relatively cheap supply of plutonium was made available. So the Manhattan Project proceeded to make another type of atomic bomb — a plutonium bomb. The plutonium bomb was made in the form of a sphere, with pieces of plutonium, each below the critical mass, at the edge of the sphere, as shown in figure 5.12. For ignition, a series of chemical explosions fired the plutonium pieces all toward the center of the sphere at the same time. When all these pieces of plutonium came together they constituted the critical mass of plutonium, and the chain reaction was initiated and the bomb exploded.

The first test of an atomic device, was a test of the plutonium bomb on July 16, 1945, at a site called "Trinity" in the New Mexico desert. The first plutonium bomb, called "Fat Boy" was dropped on Nagasaki on August 9, 1945.

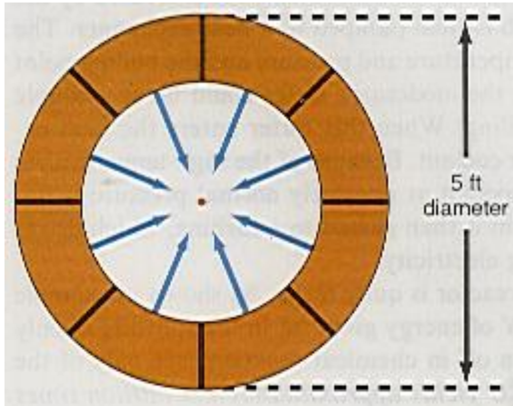
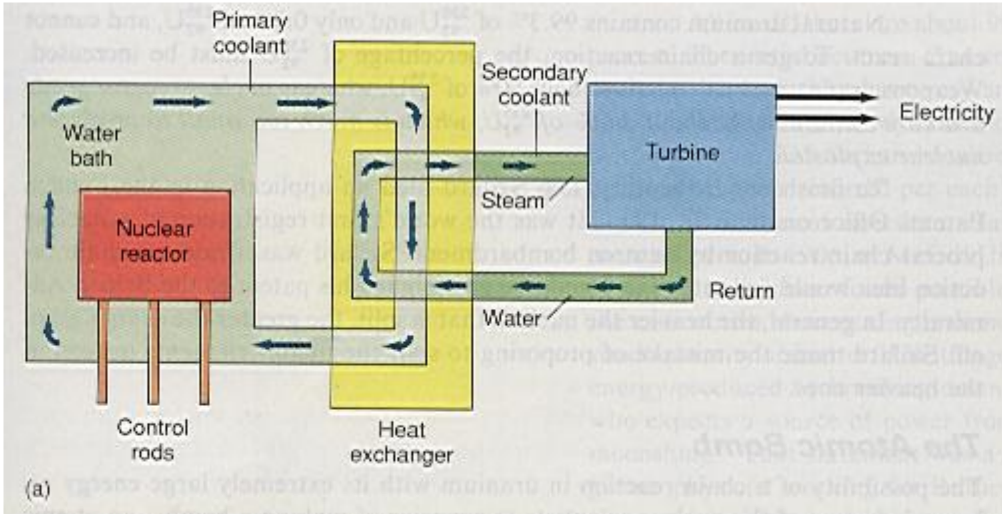


Figure 5.12 Triggering the plutonium bomb.

Fission Nuclear Reactors

The first nuclear reactor was built by Enrico Fermi on the squash court under the west stands of Stagg Field at the University of Chicago. It was started in October of 1942 and began operating on December 2, 1942. This was the first controlled use of nuclear fission.

A typical nuclear reactor is sketched in figure 5.13. The reactor itself contains uranium, ${}^{238}_{92}U$, enriched with 3.6% of ${}^{235}_{92}U$ Neutrons are given off by a reaction such as equation 5.36. The neutrons given off have a rather high kinetic energy and are called *fast neutrons* because of the high speed associated with the large kinetic energy. These fast neutrons are moving too fast to initiate more fission reactions and must be slowed down. One such way is to enclose the entire reactor in a water bath under high pressure. Such a reactor is called a pressurized water reactor (PWR). The neutrons now collide with the water molecules and are slowed down so that they can be used in the fission process. The water is called the *moderator,* because it moderates or slows down the neutrons. The slow neutrons now proceed to split more ${}^{235}_{92}U$ nuclei until a chain reaction is obtained. The chain reaction is not allowed to run wild as in an atomic bomb but is controlled by a series of rods, usually made of cadmium, that are inserted into the reactor. Cadmium is an element that is capable of absorbing a large number of neutrons without becoming unstable or radioactive. Hence, when the cadmium control rods are inserted into the reactor they absorb neutrons to cut down on the number of neutrons that are available for the fission process. In this way, the fission reaction is controlled. The water moderator also acts as a coolant. The tremendous heat generated by the fission process heats up the water, which is then pumped to a heat exchanger. The hot moderator water is at a very high temperature and pressure, and the boiling point of water increases with pressure. Thus, the moderator water could be at a couple of hundred degrees Celsius without boiling. When this water enters the heat exchanger, it heats up the secondary water coolant. Because of the high temperatures of the primary coolant, the secondary coolant at relatively normal



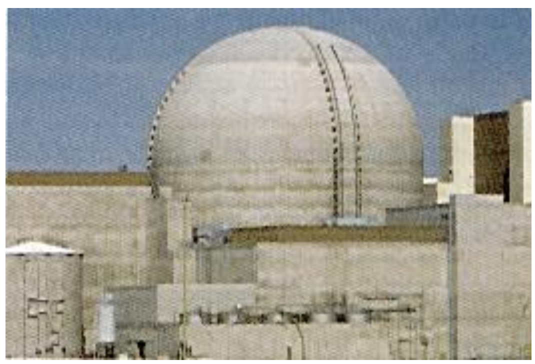


Figure 5.13 A typical nuclear reactor.

pressure is immediately converted to steam. This steam is then passed to a turbine, which drives an electric generator, thereby producing electricity.

The energy that comes from a reactor is quite large. As shown in example 5.10, there are approximately 200 MeV of energy given off in the splitting of only one ${}^{235}_{92}U$ nucleus. Typical energies given off in chemical reactions are only of the order of 3 or 4 eV. Hence, *fission of* ${}^{235}_{92}U$ yields approximately 2.5 million times as much energy as found in the combination of the same mass of carbon (such as in coal or gasoline).

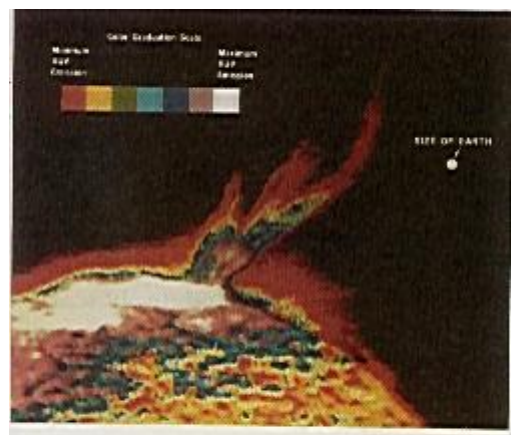
The one drawback to a fission reactor is the nuclear waste material. As shown in equations 5.36 through 5.39, fission fragments such as ${}^{141}_{56}Ba$, ${}^{92}_{36}Kr$, ${}^{144}_{56}Ba$, ${}^{89}_{36}Kr$, ${}^{140}_{54}Xe$, ${}^{94}_{38}Sr$, ${}^{132}_{50}Sn$, and ${}^{101}_{42}Mo$, are some of the possible products of the reaction. These isotopes are unstable and decay into other radioactive nuclei. Eventually all these dangerous radioactive waste nuclei must be discarded. Some of them have relatively long half-lives and will, therefore, be around for a long time. They cannot be dumped into oceans or left in any place where they will contaminate the environment, such as through the soil or the air. They must not be allowed to get into the drinking water. The best place so far found to store these wastes is in the bottom of old salt mines, which are very dry and are thousands of feet below the surface of the earth. Here they can sit and decay without polluting the environment.

One unfounded fear of many people is that a nuclear reactor may explode like an atomic bomb and kill all the people in its neighborhood. A nuclear reactor does not contain enough $^{235}_{92}U$ to explode as an atomic bomb. What is more, the cadmium control rod's normal position is in the reactor. They must be pulled out to get and keep the reactor in operation. Any failure of any mechanism of the reactor causes the control rods to fall back into the reactor, thereby, stopping the chain reaction and shutting down the reactor.

Another type of a fission nuclear reactor is the *breeder reactor*. A breeder reactor uses uranium $^{238}_{92}U$, or thorium $^{232}_{90}Th$, as the nuclear fuel and uses fast high-energy neutrons instead of the slow ones used in the PWR. The fast neutrons react with the $^{238}_{92}U$, according to equation 5.28, and form neptunium, $^{239}_{93}Np$ The neptunium, $^{239}_{93}Np$, decays according to equation 5.29, and produces plutonium, $^{239}_{94}Pu$ The plutonium is highly fissionable and it too can supply energy in the reactor. The net result of forming plutonium in the reactor is to create more fissionable material than is used. Hence, the name breeder reactor; it "breeds" nuclear fuel. Of course, the breeder reactor can also generate electricity while it is creating more fuel. Breeder reactors are used to create plutonium for nuclear weapons.

5.8 Nuclear Fusion

It has been long observed that the sun emits tremendous quantities of energy for an enormous quantity of time. There was much speculation as to the source of this energy. In 1938, Hans Bethe (1906-) suggested that the fusion of hydrogen nuclei into helium nuclei was responsible for the tremendous energy released. ***Nuclear fusion*** *is a process in which lighter nuclei are joined together to produce a heavier nucleus and a good deal of energy*. Bethe proposed that the energy was released in the sun in what he called the proton-proton cycle. The first part of the cycle consists of two protons combining to form an unstable isotope of helium.



The enormous energy generated by stars like our Sun is a result of nuclear fusion.

$$^{1}_{1}p + ^{1}_{1}p \rightarrow ^{2}_{2}He$$

But one of these combined protons in the nucleus of the unstable isotope immediately decays by equation 5.21 as

$$^{1}_{1}p \rightarrow ^{1}_{0}n + ^{0}_{+1}e + \nu$$

The neutron now combines with the first proton to form the deuteron, and we can write the entire reaction as

$$ {}^{1}_{1}\text{p} + {}^{1}_{1}\text{p} \rightarrow {}^{2}_{1}\text{H} + {}^{0}_{+1}\text{e} + \nu \tag{5.40}$$

Note here that the decay of a proton or a neutron in the nucleus is caused by the weak nuclear force, which we will describe in more detail in chapter 6. The deuteron formed in equation 5.40 now combines with another proton to form the isotope of helium, 3 2 He, according to the reaction

$$ {}^{2}_{1}\text{H} + {}^{1}_{1}\text{p} \rightarrow {}^{3}_{2}\text{He} + \gamma \tag{5.41}$$

The process represented by equations 5.40 and 5.41 must occur twice to form two ${}^{3}_{2}\text{He}$ nuclei, which then react according to the equation

$$ {}^{3}_{2}\text{He} + {}^{3}_{2}\text{He} \rightarrow {}^{4}_{2}\text{He} + 2\ {}^{1}_{1}\text{p} \tag{5.42}$$

Thus, the stable element helium has been formed from the fusion of the nuclei of the hydrogen atom. We can write the entire proton-proton cycle in the shorthand version as

$$ 4\ {}^{1}_{1}\text{p} \rightarrow {}^{4}_{2}\text{He} + 2\ {}^{0}_{+1}\text{e} + 2\gamma + 2\nu + \boldsymbol{Q} \tag{5.43}$$

The net $\boldsymbol{Q}$ value, or energy released in the process, is about 26 MeV.

The Hydrogen Bomb

In July of 1942, Robert Oppenheimer (1904-1967), reporting on the work of Edward Teller, Enrico Fermi, and Hans Bethe, noted that the extremely high temperature of an atomic bomb could be used to trigger a fusion reaction in deuterium, thus producing a fusion bomb or a hydrogen bomb. The reaction between deuterium and tritium, both isotopes of hydrogen, is given by

$$ {}^{2}_{1}\text{H} + {}^{3}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He} + {}^{1}_{0}\text{n} + 17.6\ \text{MeV}$$

Deuterium is relatively abundant in ocean water but tritium is relatively scarce. However, tritium can be generated in a nuclear reactor by surrounding the core with lithium. The neutron from the reactor causes the reaction

$$ {}^{7}_{3}\text{Li} + {}^{1}_{0}\text{n} \rightarrow {}^{4}_{2}\text{He} + {}^{3}_{1}\text{H} + {}^{1}_{0}\text{n}$$

Thus, all the tritium desired can be relatively easily created.

The hydrogen bomb is effectively a bomb within a bomb, as illustrated in figure 5.14. A conventional atomic bomb made of plutonium is ignited. The tremendous heat given off by the A-bomb supplies the high temperature to start the fusion process of the deuterium-tritium mixture. The size of an A-bomb is limited by the critical mass of plutonium. We cannot assemble an amount of plutonium

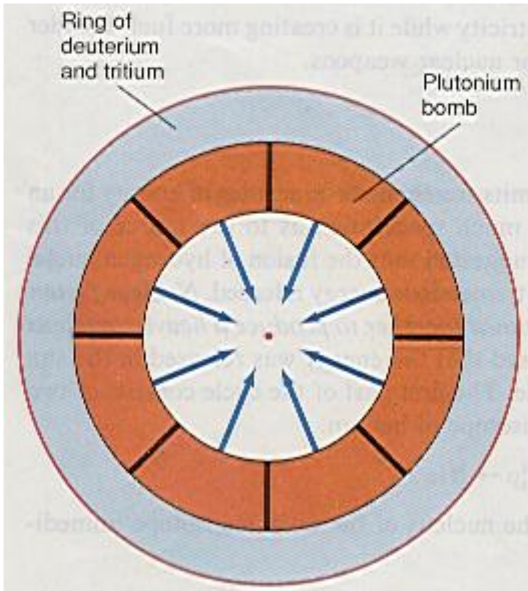


Figure 5.14 A hydrogen bomb. The energy released in an explosion resulting from the nuclear fusion of even small amounts of material is enormous.

greater than the critical mass without it exploding. We can, however, assemble as much deuterium and tritium as we please. It will never go off unless supplied with the extremely high temperature necessary for fusion.

The first H-bomb was detonated on October 31, 1952. It completely eliminated the island of Eniwetok in the Marshall Islands. The Soviet Union quickly followed suit by exploding their H-Bomb on August 12, 1953. The Soviets used lithium in place of tritium in their fusion reaction, because it is cheaper and more easily available.

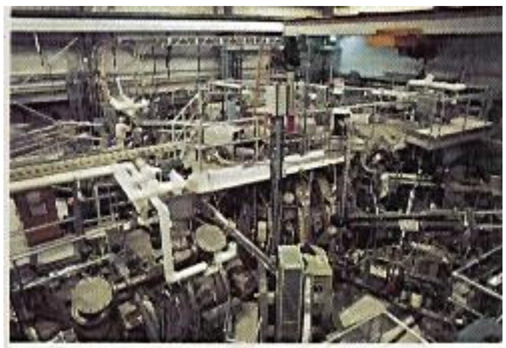
The Fusion Reactor

One of the difficulties of a fission reactor is the radioactive fragments or waste that is a by-product of the reaction. *The fusion process has no by-product that is radioactive.* That is, the only result of fusion, is helium, which is an inert gas and is not radioactive. The proton-proton cycle in the sun is too slow to take place in a reactor. Hence, the fusion cycle in a fusion reactor is given by

$$ {}_1^2\mathrm{H} + {}_1^2\mathrm{H} \rightarrow {}_1^3\mathrm{H} + {}_1^1\mathrm{p} $$
$$ {}_1^2\mathrm{H} + {}_1^3\mathrm{H} \rightarrow {}_2^4\mathrm{He} + {}_0^1\mathrm{n} $$

The difficulty in the design of a fusion reactor has to do with the extremely high temperatures associated with the fusion process, that is, millions of kelvins. (Remember the surface temperature of the sun is about 6000 K, and the core, thousands of times higher.) At these high temperatures, all materials that could be made to contain the reaction would melt. The task of building a fusion reactor is not, however, impossible, just difficult. At the high temperatures of fusion, electrons and nuclei are completely separated from each other in what is called a *plasma,* an ionized fluid. Because of the electric charges of the fluid, the fusion reaction can be contained within magnetic fields. Experimental fusion reactors have been built on a very limited scale using magnetic confinement with some slight success. A great deal of work still has to be done to perfect the fusion reactor. This work must be done, because the fusion reactor promises to be a source of enormous energy produced by a very cheap fuel, effectively water, with no radioactive contaminants as a by-product.

A tokamak apparatus, designed to generate the magnetic field necessary for containing a fusion reaction.



Thermonuclear fusion reactor.

5.9 Nucleosynthesis

It is a fact of life that we all take for granted the things that are around us. On this planet earth, the materials we see are made out of molecules and atoms. We saw in the discussion of the periodic table of the elements in chapter 4 how each element differs from each other by the number of electrons, protons, and neutrons contained within each atom. *But what is the origin of all these elements? How were they originally formed? The elements found on the earth and throughout the universe were originally synthesized by the process of fusion within the stars, a process called* ***nucleosynthesis.*** The proton-proton cycle formed helium from hydrogen. As a continuation of that cycle, ${}^{3}_{2}\text{He}$ can fuse with ${}^{4}_{2}\text{He}$ to produce beryllium according to the equation

$${}^{3}_{2}\text{He} + {}^{4}_{2}\text{He} \rightarrow {}^{7}_{4}\text{Be} + \gamma$$

Beryllium can now capture an electron to form lithium according to the relation

$${}^{7}_{4}\text{Be} + {}^{0}_{-1}\text{e} \rightarrow {}^{7}_{3}\text{Li} + \nu$$

As most of the hydrogen is used, the helium nuclei can now fuse to form the following nuclei

$$
\begin{gathered}
{}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{8}_{4}\mathrm{Be} + \gamma \\
{}^{8}_{4}\mathrm{Be} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{12}_{6}\mathrm{O}^{*} \rightarrow {}^{12}_{6}\mathrm{C} + 2\gamma \\
{}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{16}_{8}\mathrm{O} + \gamma \\
{}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{20}_{10}\mathrm{Ne} + \gamma
\end{gathered}
$$

If the star continues burning, additional elements are formed, such as

$$
\begin{gathered}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} \rightarrow {}^{13}_{6}\mathrm{C} + {}^{0}_{+1}\mathrm{e} + \nu \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} \rightarrow {}^{15}_{7}\mathrm{N} + {}^{0}_{+1}\mathrm{e} + \nu \\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{16}_{8}\mathrm{O} + \gamma \\
{}^{16}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{p} \rightarrow {}^{17}_{9}\mathrm{F} + \gamma
\end{gathered}
$$

As the nuclear fusion process continues, all the chemical elements and their isotopes up to about an atomic number of 56 or so, are created within the stars. If the star is large enough it eventually explodes as a supernova, spewing its contents into interstellar space. It is believed that the high temperatures in the explosion cause the formation of the higher chemical elements above iron. Some of the dust from these clouds is gradually pulled together by gravity to form still new stars. *Hence, stars are factories for the creation of the chemical elements.* If some of these fragments of the supernova are caught up in the gravitational field of another star, they could, with the correct initial velocity, go into orbit around the new star. The captured fragments of the star would slowly condense and become a planet with a complete set of elements as are now found on the planet earth.

Have you ever wondered … ?
An Essay on the Application of Physics
Radioactive Dating

Have you ever wondered how scientists are able to determine the age of very old objects? The technique used to determine their age is called **radioactive dating** and it is based upon the amount of unstable isotopes still contained in them. Perhaps the most famous of these techniques is *carbon dating.* Cosmic rays, which are high-energy protons and neutrons from outer space, impinge on the earth's upper atmosphere, and cause nuclear reactions with the nitrogen present there. The result of these nuclear reactions is to create an unstable isotope of carbon, namely, ${}^{14}_{6}\mathrm{C}$, which has a half-life of 5770 yr. It is assumed that the total amount of this isotope remains constant with time because of an equilibrium between the amount being formed at any time and the amount decaying at any time. This isotope of carbon combines chemically with the oxygen, O_2, in the atmosphere to form carbon dioxide, CO_2. Most of the carbon dioxide in the atmosphere is, of course, formed from ordinary carbon, ${}^{12}_{6}C$ Because the chemical properties depend

Figure 5H.1 A fossil of a seed fern. How can you tell how old it is?

on the orbital electrons and not the nucleus, ${}^{14}_{6}C$ reacts chemically the same as ${}^{12}_{6}C$ Hence, we cannot determine chemically whether the carbon dioxide is made from carbon ${}^{12}_{6}C$ or carbon ${}^{14}_{6}C$.

The green plants in the environment convert water, H_2O, and carbon dioxide, CO_2, into carbohydrates by the process of photosynthesis. Hence, the radioactive isotope ${}^{14}_{6}C$ becomes a part of every living plant. Animals and humans eat these plants while also exhaling carbon dioxide. Thus plants, animals, and humans are found to contain the radioactive isotope ${}^{14}_{6}C$ The ratio of the carbon isotope ${}^{14}_{6}C$ to ordinary carbon ${}^{12}_{6}C$ is a constant in the atmosphere and all living things. The ratio is of course quite small, approximately 1.3×10^{-12}. That is, the amount of carbon ${}^{14}_{6}C$ is equal to 0.0000000000013 times the amount of ordinary carbon. Whenever any living thing dies, the radioactive isotope ${}^{14}_{6}C$ is no longer replenished and decreases by beta decay according to the reaction

$${}^{14}_{6}C \rightarrow {}^{14}_{7}N + {}^{0}_{-1}e + \bar{\nu} \tag{5H.1}$$

Thus, the ratio of ${}^{14}_{6}C/{}^{12}_{6}C$ is no longer a constant, but starts to decay with time. Thus, by knowing the present ratio of ${}^{14}_{6}C/{}^{12}_{6}C$, the age of the particular object can be determined. In practice, the amount of ${}^{14}_{6}C$ nuclei is relatively difficult to measure, whereas its activity, the number of disintegrations per unit time, is not. Using equation 5.12 for the activity of a radioactive nucleus, we get

$$\frac{A}{A_0} = e^{-\lambda t}$$

Taking the natural logarithms of both sides of the equation, we get

$$\ln\left(\frac{A}{A_0}\right) = -\lambda t$$

Solving for the time t, we get

$$t = \frac{-\ln(A/A_0)}{\lambda} \tag{5H.2}$$

Thus, if the activity A_0 of a present living thing is known, and the activity A of the object we wish to date is measured, we can solve equation 5H.2 for its age.

Example 5H.1

Carbon dating. A piece of wood believed to be from an ancient Egyptian tomb is tested in the laboratory for its carbon-14 activity. It is found that the old wood has an activity of 10.0 disintegrations/min, whereas a new piece of wood has an activity of 15.0 disintegrations/min. Find the age of the wood.

Solution

First, we find the decay constant of ${}^{14}_{6}C$ from equation 5.11 as

$$\lambda = \frac{0.693}{T_{1/2}}$$
$$= \left(\frac{0.693}{5770\text{ yr}}\right)\left(\frac{1\text{ yr}}{3.1535\times 10^7\text{ s}}\right)$$
$$= 3.81 \times 10^{-12}\text{ /s}$$

We find the age of the wood from equation 5H.2 with A = 10.0 disintegrations/min and A_0 = 15 disintegrations/min. Thus,

$$t = \frac{-\ln(A/A_0)}{\lambda}$$
$$t = \frac{-\ln(10.0/15.0)}{3.81\times 10^{-12}/\text{s}}$$
$$= (1.06\times 10^{11}\text{ s})\left(\frac{1\text{ yr}}{3.1535\times 10^7\text{ s}}\right)$$
$$= 3370\text{ yr}$$

Hence, the wood must be 3370 yr old.

Go to Interactive Example

Similar dating techniques are used in geology to determine the age of rocks. As an example, the uranium atom ${}^{238}_{92}U$ decays through a series of steps and ends up as the stable isotope of lead, ${}^{206}_{82}Pb$ The ratio of the abundance of ${}^{238}_{92}U$ to ${}^{206}_{82}Pb$ can be used to determine the age of a rock.

The Language of Physics

Atomic number *Z*
The number of protons or electrons in an atom (p.).

Mass number *A*
The number of protons plus neutrons in the nucleus (p.).

Neutron number *N*
The number of neutrons in the nucleus. It is equal to the difference between the mass number and the atomic number (p.).

Isotope
An isotope of a chemical element has the same number of protons as the element but a different number of neutrons. An isotope reacts chemically in the same way as the parent element. Its observable difference is its different atomic mass, which comes from the excess or deficiency of neutrons in the nucleus (p.).

Atomic mass
The mass of a chemical element that is listed in the periodic table of the elements. That atomic mass is an average of the masses of its different isotopes (p.).

Strong nuclear force
The force that binds protons and neutrons together in the nucleus. Whenever the nuclear force is less than the electrostatic force, the nucleus breaks up or decays, and emits radioactive particles (p.).

Mass defect
The difference in mass between the sum of the masses of the constituents of a nucleus and the mass of the nucleus (p.).

Binding energy
The energy that binds the nucleus together. It is the mass defect expressed as an energy (p.).

Radioactivity
The spontaneous disintegration of the nuclei of an atom with the emission of α, β, or γ particles (p.).

Activity
The rate at which nuclei decay with time (p.).

Half-life
The time it takes for half the original radioactive nuclei to decay (p.).

Alpha decay
A disintegration of an atomic nucleus whereby an α particle is emitted. The original element of atomic number Z is transmuted into a new chemical element of atomic number $Z - 2$ (p.).

Beta decay, β^-
A nuclear decay whereby a neutron within the nucleus decays into a proton, an electron, and an antineutrino. The proton stays in the nucleus, but the electron and antineutrino are emitted. Thus, the atomic number Z increases by 1, but the mass number A stays the same. Hence, a chemical element Z is transmuted into the element $Z + 1$ (p.).

Beta decay, β^+
A nuclear decay whereby a proton within the nucleus decays into a neutron, a positron, and a neutrino. The positron and neutrino are emitted but the neutron stays behind in the nucleus. The atomic number Z of the element decreases by one because of the loss of the proton. Hence, an element of atomic number Z is converted into the element $Z - 1$ (p.).

***Q* value of a nuclear reaction**
The energy available in a reaction caused by the difference in mass between the reactants and the products (p.).

Exoergic reaction
A nuclear reaction in which energy is released. It is sometimes called an exothermic reaction (p.).

Endoergic reaction
A nuclear reaction in which energy must be added to the system to make the reaction proceed. It is sometimes called an endothermic reaction (p.).

Nuclear fission
The process of splitting a heavy atom into two lighter atoms (p.).

Nuclear fusion
The process in which lighter nuclei are joined together to produce a heavier nucleus with a large amount of energy released (p.).

Nucleosynthesis
The formation of the nuclei of all the chemical elements by the process of fusion within the stars (p.).

Radioactive dating
A technique in which the age of very old objects can be determined by the amount of unstable isotopes still contained in them (p.).

Summary of Important Equations

Neutron number $$N = A - Z \quad (5.1)$$

Representation of a nucleus $$^{A}_{Z}X \quad (5.2)$$

Mass defect $$\Delta m = Zm_{\mathbf{p}} + (A - Z)m_{\mathbf{n}} - m_{\text{nucleus}}c^2 \quad (5.3)$$

Binding energy $$\text{BE} = (\Delta m)c^2 = Zm_p c^2 + (A - Z)m_n c^2 - m_{nucleus}c^2 \quad (5.5)$$

Rate of nuclear decay $$\frac{dN}{dt} = -\lambda N \quad (5.6)$$

Activity $$A = -\frac{dN}{dt} = \lambda N \quad (5.7)$$

Radioactive decay law $$N = N_0 e^{-\lambda t} \quad (5.8)$$

Decay constant $$\lambda = \frac{0.693}{T_{1/2}} \quad (5.11)$$

Alpha decay $$^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}X + {}^{4}_{2}\text{He} \quad (5.16)$$

Neutron decay $$^{1}_{0}\text{n} \rightarrow {}^{1}_{1}\text{p} + {}^{0}_{-1}\text{e} + \bar{\nu} \quad (5.19)$$

Beta– decay $$^{A}_{Z}X \rightarrow {}^{A}_{Z+1}X + {}^{0}_{-1}\text{e} + \bar{\nu} \quad (5.20)$$

Proton decay $$^{1}_{1}\text{p} \rightarrow {}^{1}_{0}\text{n} + {}^{0}_{+1}\text{e} + \nu \quad (5.21)$$

Beta⁺ decay $$^{A}_{Z}X \rightarrow {}^{A}_{Z-1}X + {}^{0}_{+1}\text{e} + \nu \quad (5.22)$$

Electron capture $$^{0}_{-1}\text{e} + {}^{1}_{1}\text{p} \rightarrow {}^{1}_{0}\text{n} + \nu \quad (5.23)$$

Electron capture $$^{0}_{-1}\text{e} + {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}X + \nu \quad (5.24)$$

Gamma decay $$^{A}_{Z}X^{*} \rightarrow {}^{A}_{Z}X + \gamma \quad (5.25)$$

Q value of a nuclear reaction
$$Q = (m_x + M_X)c^2 - (m_y + M_Y)c^2 \quad (5.31)$$
$$Q = [(\text{Input mass}) - (\text{Output mass})]c^2 \quad (5.32)$$
$$Q = E_{in} - E_{out} \quad (5.33)$$

General form of equation for nuclear reaction $$x + X = y + Y + Q \quad (5.34)$$

Nuclear fission of $^{235}_{92}\text{U}$ $$^{1}_{0}\text{n} + {}^{235}_{92}\text{U} \rightarrow y + Y + {}^{1}_{0}\text{n} + Q \quad (5.35)$$

Proton-proton cycle of nuclear fusion
$$^{1}_{1}\text{p} + {}^{1}_{1}\text{p} \rightarrow {}^{2}_{1}\text{H} + {}^{0}_{+1}\text{e} + \nu \quad (5.40)$$
$$^{2}_{1}\text{H} + {}^{1}_{1}\text{p} \rightarrow {}^{3}_{2}\text{He} + \gamma \quad (5.41)$$
$$^{3}_{2}\text{He} + {}^{3}_{2}\text{He} \rightarrow {}^{4}_{2}\text{He} + 2\,{}^{1}_{1}\text{p} \quad (5.42)$$

Radioactive age $$t = \frac{-\ln(A/A_0)}{\lambda} \quad (5H.2)$$

Questions for Chapter 5

1. What are isotopes? What do they have in common and what are their differences?

2. What is the difference between fast neutrons and slow neutrons, and how do they have an effect on nuclear reactions?

3. What do we mean by the term critical mass?

4. Discuss the advantages and disadvantages of nuclear power compared to the use of fossil-fuel-generated power.

*5. What is a radioactive tracer and how is it used in medicine?

6. Explain the difference between nuclear fission and nuclear fusion.

7. Should an atomic bomb really be called a nuclear bomb?

8. How is the half-life of a radioactive substance related to its activity?

*9. Was the Chernobyl Nuclear Reactor explosion in the Soviet Union a nuclear explosion? Does the fact that the reactor was a breeder reactor, rather than a commercial electricity generator, have anything to do with the severity of the disaster?

Problems for Chapter 5

Section 5.2 Nuclear Structure

1. Find the atomic number, the mass number, and the neutron number for (a) $^{58}_{29}\mathrm{Cu}$, (b) $^{24}_{11}\mathrm{Na}$, (c) $^{210}_{84}\mathrm{Po}$, (d) $^{45}_{20}\mathrm{Ca}$, and (e) $^{206}_{82}\mathrm{Pb}$.

2. Determine the number of protons and neutrons in one atom of (a) $^{87}_{37}\mathrm{Rb}$, (b) $^{40}_{19}\mathrm{K}$, (c) $^{137}_{55}\mathrm{Cs}$, (d) $^{60}_{27}\mathrm{Co}$, and (e) $^{131}_{53}\mathrm{I}$.

3. Find the number of protons in 1 g of $^{40}_{19}\mathrm{K}$.

4. $^{63}_{29}\mathrm{Cu}$ has an atomic mass of 62.929595 u and an abundance of 69.09%, whereas $^{65}_{29}\mathrm{Cu}$ has an atomic mass of 64.927786 u and an abundance of 30.91%. Find the atomic mass of the element copper.

5. $^{107}_{47}\mathrm{Ag}$ has an atomic mass of 106.905095 u and an abundance of 51.83%, whereas $^{109}_{47}\mathrm{Ag}$ has an atomic mass of 108.904754 u and an abundance of 48.17%. Find the atomic mass of the element silver.

6. Find the mass defect and the binding energy for the helium nucleus if the atomic mass of the helium nucleus is 4.0026 u.

7. Find the mass defect and the binding energy for tritium if the atomic mass of tritium is 3.016049 u.

8. How much energy would be released if six hydrogen atoms and six neutrons were combined to form $^{12}_{6}\mathrm{C}$?

Section 5.3 Radioactive Decay Law

9. $^{63}_{28}\mathrm{Ni}$ has a half-life of 92 yr. Find its decay constant.

10. $^{235}_{92}U$ has a half-life of 7.038×10^8 yr. Find its decay constant.

11. An unknown sample has a decay constant of 2.83×10^{-6} 1/s. Find the half-life of the sample.

12. The decay constant of $^{14}_{6}C$ is $\lambda = 3.86 \times 10^{-12}$ s^{-1}. If there are 7.35×10^{90} atoms of carbon fourteen at $t = 0$, how many of them will decay in a time of $t = 2.00 \times 10^{12}$ s?

13. A sample contains 0.200 moles of $^{65}_{30}Zn$ If $^{65}_{30}Zn$ has a decay constant of 3.27×10^{-8} /s, find the number of $^{65}_{30}Zn$ nuclei present at the end of 1 day.

14. One gram of $^{87}_{36}Kr$ has a half-life of 78.0 min. How many of these nuclei are still present at the end of 15.0 min?

15. $^{60}_{27}Co$ has a half-life of 5.27 yr. How long will it take for 90.0% of the original sample to disintegrate?

16. $^{90}_{38}Sr$ has a half-life of 28.8 yr. How long will it take for it to decay to 10.0% of its original value?

17. A dose of 1.85×10^6 Bq of radioactive iodine, $^{131}_{53}I$, is used in the treatment of a disorder of the thyroid gland. If its half-life is 8 days, find the activity after (a) 8 days, (b) 16 days, and (c) 32 days.

18. In a given sample of radioactive material, the number of original nuclei drops from 6.00×10^{50} to 1.50×10^{50} in 4.50 s. Find (a) the half-life and (b) the mean lifetime (τ_{avg}) of the material.

Section 33.4 Forms of Radioactivity

19. $^{220}_{86}Rn$ decays by alpha emission. What isotope is formed?
20. $^{230}_{90}Th$ decays by alpha emission. What isotope is formed?
21. If ^{233}U decays twice by alpha emission, what is the resulting isotope?
22. $^{214}_{84}Po$ decays by β^- decay. What isotope is formed?
23. $^{210}_{82}Pb$ decays by β^- decay. What isotope is formed?
24. $^{33}_{17}Cl$ decays by β^+ decay. What isotope is formed?
25. $^{49}_{24}Cr$ decays by β^+ decay. What isotope is formed?
26. $^{41}_{20}Ca$ decays by electron capture. What isotope is formed?
27. $^{52}_{25}Mn$ decays by electron capture. What isotope is formed?

Section 5.6 Energy in Nuclear Reactions

28. How much energy is released or absorbed in the following reaction?

$$^{216}_{84}Po \rightarrow {}^{212}_{82}Pb + {}^{4}_{2}He$$

The atomic mass of $^{216}_{84}Po$ is 216.0019 u, $^{4}_{2}He$ is 4.002603 u, and $^{212}_{82}Pb$ is 211.9919 u.

29. Determine the energy associated with the reactions

$$^{1}_{0}n \rightarrow {}^{1}_{1}p + {}^{0}_{-1}e + \bar{\nu}$$

$$ {}^{1}_{1}\mathrm{p} \to {}^{1}_{0}\mathrm{n} + {}^{0}_{+1}\mathrm{e} + \nu $$

30. Find the Q value associated with the reaction

$$ {}^{1}_{1}\mathrm{H} + {}^{14}_{7}\mathrm{N} \to {}^{15}_{8}\mathrm{O} + \nu $$

The atomic mass of ${}^{14}_{7}\mathrm{N}$ is 14.003074 and ${}^{15}_{8}\mathrm{O}$ is 15.003072 u.

31. Find the Q value associated with the reaction

$$ {}^{14}_{6}\mathrm{C} \to {}^{14}_{7}\mathrm{N} + {}^{0}_{-1}\mathrm{e} + \bar{\nu} + Q $$

32. Find the Q value associated with the nuclear fission reaction

$$ {}^{1}_{0}\mathrm{n} + {}^{235}_{92}\mathrm{U} \to {}^{132}_{50}\mathrm{Sn} + {}^{101}_{42}\mathrm{Mo} + 3\ {}^{1}_{0}\mathrm{n} + Q $$

The atomic mass of ${}^{235}_{92}\mathrm{U}$ is 235.043933 u, ${}^{132}_{50}\mathrm{Sn}$ is 49.917756 u, and ${}^{101}_{42}\mathrm{Mo}$ is 41.910346 u.

33. Find the Q value of the fusion reaction

$$ {}^{2}_{1}\mathrm{H} + {}^{3}_{1}\mathrm{H} \to {}^{4}_{2}\mathrm{He} + {}^{1}_{0}\mathrm{n} + Q $$

Additional Problems

*34. A 5.00-g sample of ${}^{60}_{27}\mathrm{Co}$ has a half-life of 5.27 yr. Find (a) the decay constant, (b) the activity of the material when $t = 0$, (c) the activity when $t = 1.00$ yr, and (d) the number of nuclei present after 1.00 yr.

*35. A 5.00-g sample of ${}^{230}_{90}\mathrm{Th}$ has a half-life of 80.0 yr, and a 5.00-g sample of ${}^{222}_{86}\mathrm{Rn}$ has a half-life of 3.82 days. For each sample find (a) the decay constant, (b) the activity of the material when $t = 0$, (c) the activity when $t = 100$ days, (d) the number of nuclei present after 100 days. (e) Comparing the activities and the number of radioactive nuclei remaining at 100 days for the two samples, what can you conclude?

36. If ${}^{231}\mathrm{Pa}$ decays first by beta decay, and then by alpha emission, what is the resulting isotope?

37. A bone from an animal is found in a very old cave. It is tested in the laboratory and it is found that it has a carbon-14 activity of 13.0 disintegrations per minute. A similar bone from a new animal is tested and found to have an activity of 25.0 disintegrations per minute. What is the age of the bone?

38. A wooden statue is observed to have a carbon fourteen activity of 7.0 disintegrations per minute. How old is the statue? (New wood was found to have an activity of 15.0 disintegrations/min.)

Interactive Tutorials

39. *Radioactive decay.* A mass of 8.55 g of the isotope ${}^{90}_{38}\mathrm{Sr}$ has a half-life $T_{1/2} =$ 28.8 yr. Find (a) the decay constant λ, (b) the number of nuclei N_0 present at the

start, (c) the activity A_0 at the start, (d) the number of nuclei N present for $t = T_{1/2}$, (e) the rate of decay of the nuclei at $t = T_{1/2}$, (f) the number of nuclei present for any time t, and (g) the activity at any time t.

Go to Interactive Tutorial

Chapter 6 Elementary Particle Physics and The Unification of The Forces

> "Three quarks for Muster Mark!
> Sure he hasn't got much of a bark
> and sure any he has it's all beside the mark"
> James Joyce, Finnegan's Wake

6.1 Introduction

Man has always searched for simplicity in nature. Recall that the ancient Greeks tried to describe the entire physical world in terms of the four quantities of earth, air, fire, and water. These, of course, have been replaced with the fundamental quantities of length, mass, charge, and time in order to describe the physical world of space, matter, and time. We have seen that space and time are not independent quantities, but rather are a manifestation of the single quantity — spacetime — and that mass and energy are interchangeable, so that energy could even be treated as one of the fundamental quantities. We also found that energy is quantized and therefore, matter should also be quantized. What is the smallest quantum of matter? That is, what are the fundamental or elementary building blocks of matter? What are the forces that act on these fundamental particles? Is it possible to combine these forces of nature into one unified force that is responsible for all the observed interactions? We shall attempt to answer these questions in this chapter.

6.2 Particles and Antiparticles

As mentioned in chapter 20, the Greek philosophers Leucippus and Democritus suggested that matter is composed of fundamental or elementary particles called atoms. The idea was placed on a scientific foundation with the publication, by John Dalton, of *A New System of Chemical Philosophy* in 1808, in which he listed about 20 chemical elements, each made up of an atom. By 1896 there were about 60 known elements. It became obvious that there must be a way to arrange these different atoms in an orderly way in order to make sense of what was quickly becoming chaos. In 1869 the Russian chemist, Dimitri Mendeleev, developed the periodic table of the elements based on the chemical properties of the elements. Order was brought to the chaos of the large diversity of elements. In fact, new chemical elements were predicted on the basis of the blank spaces found in the periodic table. Later with the discovery of the internal structure of the atom, the atom could no longer be considered as elementary

By 1932, only four elementary particles were known; the electron, the proton, the neutron, and the photon. Things looked simple again. But this simplicity was not to last. Other particles were soon discovered in cosmic rays. Cosmic rays are particles from outer space that impinge on the top of the atmosphere. Some of them make it to the surface of the earth, whereas others decay into still other particles before they reach the surface. Other new particles were found in the large accelerating machines made by man. Today, there are hundreds of such particles. Except for the electron, proton, and neutron, most of these elementary particles decay very quickly. We are again in the position of trying to make order out of the chaos of so many particles.

The first attempt at order is the classification of particles according to the scheme shown in figure 6.1. All the elementary particles can be grouped into particles called hadrons or leptons.

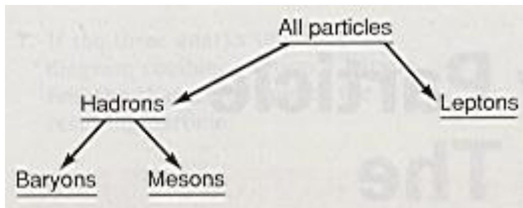


Figure 6.1 First classification of the elementary particles.

Leptons

*The **Leptons** are particles that are not affected by the strong nuclear force.* They are very small in terms of size, in that they are less than 10^{-19} m in diameter. They all have spin ½ in units of $\hbar$. There are a total of six leptons: the electron, e^-, the muon, μ^- and the tauon, τ^-, each with an associated neutrino. They can be grouped in the form

$$\begin{matrix} (\nu_e) & (\nu_\mu) & (\nu_\tau) \\ (e^-) & (\mu^-) & (\tau^-) \end{matrix} \tag{6.1}$$

There are thus three neutrinos: the neutrino associated with the electron, ν_e; the neutrino associated with the muon, ν_μ; and the neutrino associated with the tauon, ν_τ. The muon is very much like an electron but it is much heavier. It has a mass about 200 times greater than the electron. It is not stable like the electron but decays in about 10^{-6} s.

Originally the word lepton, which comes from the Greek word *leptos* meaning small or light in weight, signified that these particles were light. However, in 1975 the τ lepton was discovered and it has twice the mass of the proton. That is, the τ lepton is a heavy lepton, certainly a misnomer.

Leptons are truly elementary in that they apparently have no structure. That is, they are not composed of something still smaller. Leptons participate in the weak nuclear force, while the charged leptons, e^-, μ^-, τ^-, also participate in the electromagnetic interaction.

The muon was originally thought to be Yukawa's meson that mediated the strong nuclear force, and hence it was called a μ^- meson. This is now known to be a misnomer, since the muon is not a meson but a lepton.

Hadrons

***Hadrons** are particles that are affected by the strong nuclear force*. There are hundreds of known hadrons. Hadrons have an internal structure, composed of what appears to be truly elementary particles called quarks. The hadrons can be further broken down into two subgroups, the baryons and the mesons.

1. ***Baryons.*** *Baryons are heavy particles that, when they decay, contain at least one proton or neutron in the decay products.* The baryons have half-integral spin, that is, 1/2 $\hbar$, 3/2 $\hbar$, and so on. We will see in a moment that all *baryons are particles that are composed of three quarks.*

2. ***Mesons.*** Originally, mesons were particles of intermediate-sized mass between the electron and the proton. However many massive mesons have since been found, so the original definition is no longer appropriate. *A meson is now defined as any particle whose decay products do not include a baryon.* We will see that *mesons are particles that are composed of a quark-antiquark pair.* All mesons have integral spin, that is, 0, 1, 2, 3, and so on. The mass of the meson increases with its spin. A list of some of the elementary particles is shown in table 6.1.

Table 6.1 List of Some of the Elementary Particles		
Leptons	electron, muon, tauon, neutrinos,	e^- μ^- τ^- ν_e, ν_μ, ν_τ
Hadrons		
Baryons	proton, neutron, delta, lambda, Sigma, Hyperon, Omega	p n Δ λ Σ Λ Ω
Mesons	pi, eta, rho, omega, delta, phi	π η ρ Ω δ ϕ

In 1928, Paul Dirac merged special relativity with the quantum theory to give a relativistic theory of the electron. A surprising result of that merger was that his equations predicted two energy states for each electron. One is associated with the electron, whereas the other is associated with a particle, like the electron in every way, except that it carries a positive charge. This new particle was called the *antielectron* or the *positron*. This was the first prediction of the existence of antimatter. The positron was found in 1932.

For every particle in nature there is associated an ***antiparticle****. The antiparticle of the proton is the antiproton.* It has all the characteristics of the proton except that it carries a negative charge. Some purely neutral particles such as the photon and the π^0 meson are their own antiparticles. Antiparticles are written with a bar over the symbol for the particle. Hence, $\bar{p}$ is an antiproton and $\bar{n}$ is an antineutron.

Matter consists of electrons, protons and neutrons, whereas ***antimatter*** *consists of antielectrons (positrons), antiprotons, and antineutrons.* Figure 6.2 shows atoms of matter and antimatter. The same electric forces that hold matter

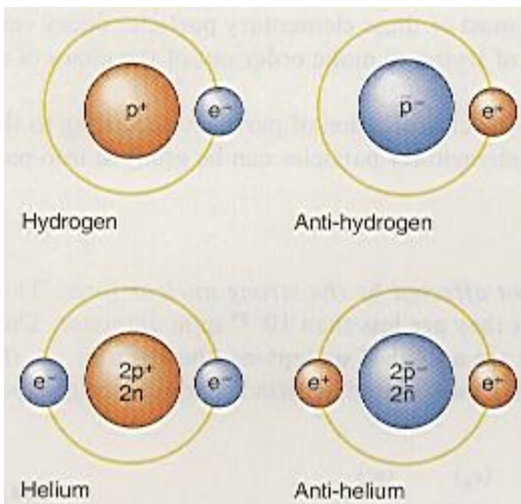


Figure 6.2 Matter and antimatter.

together, hold antimatter together. (Note that the positive and negative signs are changed in antimatter.) The antihelium nucleus has already been made in high-energy accelerators.

Whenever particles and antiparticles come together they annihilate each other and only energy is left. For example, when an electron comes in contact with a positron they annihilate according to the reaction

$$e^- + e^+ \rightarrow 2\gamma \tag{6.2}$$

where the 2γ's are photons of electromagnetic energy. (Two gamma rays are necessary in order to conserve energy and momentum.) This energy can also be used to create other particles. Conversely, particles can be created by converting the energy in the photon to a particle-antiparticle pair such as

$$\gamma \rightarrow e^- + e^+ \tag{6.3}$$

Creation or annihilation can be shown on a spacetime diagram, called a *Feynman diagram*, after the American physicist Richard Feynman (1918-1988), such as in figure 6.3. Figure 6.3(a) shows the creation of an electron-positron pair. A

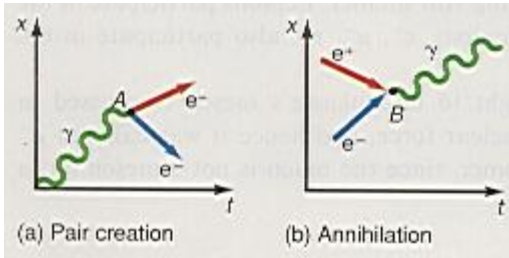


Figure 6.3 Creation and annihilation of particles.

photon γ moves through spacetime until it reaches the spacetime point *A,* where the energy of the photon is converted into the electron-positron pair. Figure 6.3(b) shows an electron and positron colliding at the spacetime point *B* where they annihilate each other and only the photon γ now moves through spacetime. (In order to conserve momentum and energy in the creation process, the presence of a relatively heavy nucleus is required.)

6.3 The Four Forces of Nature

In the study of nature, four forces that act on the particles of matter are known. They are:

1. *The Gravitational Force.* The gravitational force is the oldest known force. It holds us to the surface of the earth and holds the entire universe together. It is a long-range force, varying as $1/r^2$. Compared to the other forces of nature it is by far the weakest force of all.
2. *The Electromagnetic Force.* The electromagnetic force was the second force known. In fact, it was originally two forces, the electric force and the magnetic force, until the first unification of the forces tied them together as a single electromagnetic force. The electromagnetic force holds atoms, molecules, solids, and liquids together. Like gravity, it is a long-range force varying as $1/r^2$.
3. *The Weak Nuclear Force.* The weak nuclear force manifests itself not so much in holding matter together, but in allowing it to disintegrate, such as in the decay of the neutron and the proton. The weak force is responsible for the fusion process occurring in the sun by allowing a proton to decay into a neutron such as given in equation 5.21. The proton-proton cycle then continues until helium is formed and large quantities of energy are given off. The nucleosynthesis of the chemical elements also occurred because of the weak force. Unlike the gravitational and electromagnetic force, the weak nuclear force is a very short range force.
4. *The Strong Nuclear Force.* The strong nuclear force is responsible for holding the nucleus together. It is the strongest of all the forces but is a very short range force.

That is, its effects occur within a distance of about 10^{-15} m, the diameter of the nucleus. At distances greater than this, there is no evidence whatsoever for its very existence. The strong nuclear force acts only on the hadrons.

Why should there be four forces in nature? Einstein, after unifying space and time into spacetime, tried to unify the gravitational force and the electromagnetic force into a single force. Although he spent a lifetime trying, he did not succeed. The hope of a unification of the forces has not died, however. In fact, we will see shortly that the electromagnetic force and the weak nuclear force have already been unified theoretically into the electroweak force by Glashow, Weinberg, and Salam, and experimentally confirmed by Rubbia. A grand unification between the electroweak and the strong force has been proposed. Finally an attempt to unify all the four forces into one superforce is presently underway.

6.4 Quarks

In the attempt to make order out of the very large number of elementary particles, Murray Gell-Mann and George Zweig in 1964, independently proposed that the hadrons were not elementary particles but rather were made of still more elementary particles. Gell-Mann called these particles, **quarks.** He initially assumed there were only three such quarks, but with time the number has increased to six. The six quarks are shown in table 6.2. *The names of the quarks are: up, down, strange, charmed, bottom, and top.* One of the characteristics of these

Table 6.2
The Quarks

Name (Flavor)	Symbol	Charge	Spin
up	u	2/3	1/2
down	d	−1/3	1/2
strange	s	−1/3	1/2
charmed	c	2/3	1/2
bottom	b	−1/3	1/2
top	t	2/3	1/2

quarks is that they have fractional electric charges. That is, the up, charmed, and top quark has 2/3 of the charge found on the proton, whereas the down, strange, and bottom quark has 1/3 of the charge found on the electron. They all have spin 1/2, in units of $\hbar$. Each quark has an antiquark, which is the same as the original quark except it has an opposite charge. The antiquark is written with a bar over the letter, that is $\bar{q}$.

We will now see that all of the hadrons are made up of quarks. The baryons are made up of three quarks:

$$\text{Baryon} = qqq \tag{6.4}$$

While the mesons are made up of a quark-antiquark pair:

$$\text{Meson} = q\bar{q} \tag{6.5}$$

As an example of the formation of a baryon from quarks, consider the proton. The proton consists of two up quarks and one down quark, as shown in figure 6.4(a).

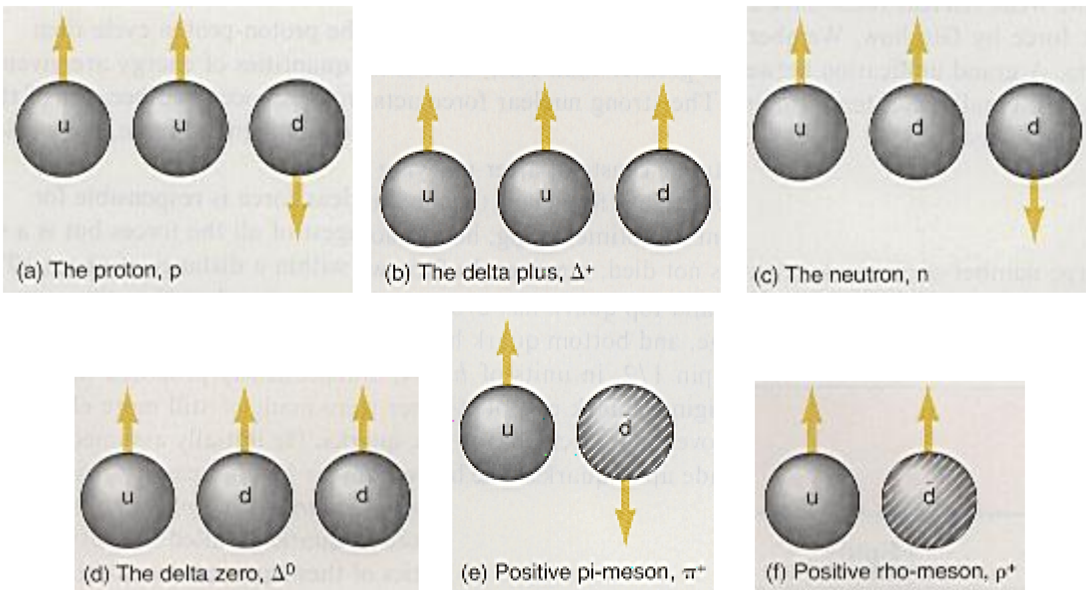


Figure 6.4 Some quark configurations of baryons and mesons.

The electric charge of the proton is found by adding the charges of the constitutive quarks. That is, since the u quark has a charge of 2/3, and the d quark has a charge of −1/3, the charge of the proton is

$$2/3 + 2/3 - 1/3 = 1$$

which is exactly as expected. Now the proton should have a spin of 1/2 in units of $\hbar$. In figure 6.4(a), we see the two up quarks as having their spin up by the direction of the arrow on the quark. The down quark has its arrow pointing down to signify that its spin is down. Because each quark has spin 1/2, the spin of the proton is found by adding the spins of the quarks as

$$1/2 + 1/2 - 1/2 = 1/2$$

We should note that the names up and down for the quarks are just that, a name, and have nothing to do with the direction of the spin of the quark. For example, the delta plus Δ^+ baryon is made from the same three quarks as the proton, but their spins are all aligned in the same direction, as shown in figure 6.4(b). Thus, the spin of the Δ^+ particle is

$$1/2 + 1/2 + 1/2 = 3/2$$

That is, the Δ^+ particle has a spin of 3/2. Since it takes more energy to align the spins in the same direction, when quark spins are aligned, they have more energy. This manifests itself as an increased mass by Einstein's equivalence of mass and energy ($E = mc^2$). Thus, we see that the mass of the Δ^+ particle has a larger mass than the proton. Hence, in the formation of particles from quarks, we not only have to know the types of quarks making up the particle but we must also know the direction of their spin.

Figure 6.4(c) shows that a neutron is made up of one up quark and two down quarks. The total electric charge is

$$2/3 - 1/3 - 1/3 = 0$$

While its spin is

$$1/2 + 1/2 - 1/2 = 1/2$$

Again note that the delta zero Δ^0 particle is made up of the same three quarks, figure 6.4(d), but their spins are all aligned.

As an example of the formation of a meson from quarks, consider the pi plus π^+ meson in figure 6.4(e). It consists of an up quark and an antidown quark. Its charge is found as

$$2/3 + [-(-1/3)] = 2/3 + 1/3 = 1$$

That is, the d quark has a charge of $-1/3$, so its antiquark $\bar{d}$ has the same charge but of opposite sign $+1/3$. The spin of the π^+ is

$$1/2 - 1/2 = 0$$

Thus, the π^+ meson has a charge of +1 and a spin of zero.

If the spins of these same two quarks are aligned, as in figure 6.4(f), the meson is the positive rho-meson ρ^+, with electric charge of +1 and spin of 1.

The quark structure of some of the baryons is shown in table 6.3, whereas table 6.4 shows the quark structure for some mesons.

Particles that contain the strange quark are called strange particles. The reason for this name is because these particles took so much longer to decay than the other elementary particles, that it was considered strange.

If a proton or neutron consists of quarks, we would like to "see" them. Just as Rutherford "saw" inside the atom by bombarding it with alpha particles, we can "see" inside a proton by bombarding it with electrons or neutrinos. In 1969, at the Stanford Linear Accelerator Center (SLAC), protons were bombarded by high-energy electrons. *It was found that some of these electrons were scattered at very large angles, just as in Rutherford scattering, indicating that there are small constituents within the proton.* Figure 6.5 shows the picture of a proton as

Table 6.3
Quark Structure of Some of the Baryons

Name	Symbol	Structure	Charge (units of e)	Spin (units of $\hbar$)	Mass (GeV)
Proton	p	u u d	1	1/2	0.938
Neutron	n	u d d	0	1/2	0.94
Delta plus plus	Δ^{++}	u u u	2	3/2	1.232
Delta plus	Δ^{+}	u u d	1	3/2	
Delta zero	Δ^{0}	u d d	0	3/2	
Delta minus	Δ^{-}	d d d	–1	3/2	
Lambda zero	Λ^{0}	u d s	0	1/2	1.116
Positive sigma	Σ^{*+}	u u s	1	3/2	1.385
Positive sigma	Σ^{+}	u u s	1	1/2	1.189
Neutral sigma	Σ^{*0}	u d s	0	3/2	1.385
Neutral sigma	Σ^{0}	u d s	0	1/2	1.192
Negative sigma	Σ^{*-}	d d s	–1	3/2	1.385
Negative sigma	Σ^{-}	d d s	–1	1/2	1.197
Negative xi	Ξ^{-}	s d s	–1	1/2	1.321
Neutral xi	Ξ^{o}	s u s	0	1/2	1.315
Omega minus	Ω^{-}	s s s	–1	3/2	1.672
Charmed lambda	Λ^{+}_{c}	u d c	1	1/2	2.281

Table 6.4
Quark Structure of Some Mesons

Name	Symbol	Structure	Charge (units of e)	Spin (units of $\hbar$)	Mass (GeV)
Positive pion	π^{+}	$\underline{d}$ u	1	0	0.14
Positive rho	ρ^{+}	$\underline{d}$ u	1	1	0.77
Negative pion	π^{-}	$\underline{u}$ d	–1	0	0.14
Negative rho	ρ^{-}	$\underline{u}$ d	–1	1	0.77
Pi zero	π^{0}	50%($\underline{u}$ u) + 50%$\underline{d}$ d)	0	0	0.135
Positive kaon	K^{+}	u $\underline{s}$	1	0	0.494
Neutral kaon	K^{0}	$\underline{s}$ d	0	0	0.498
Negative kaon	K^{-}	$\underline{u}$ s	–1	0	0.494
J/Psi (charmonium)	J/Ψ	c $\underline{d}$	0	1	3.097
Charmed eta	η_{c}	c $\underline{c}$	0	0	2.98
Neutral D	D^{0}	$\underline{u}$ c	0	0	1.863
Neutral D	D^{*0}	$\underline{u}$ c	0	1	
Positive D	D^{+}	$\underline{d}$ c	1	0	1.868
Zero B-meson	B^{0}	$\underline{d}$ b	0		5.26
Negative B-meson	B^{-}	$\underline{u}$ b	–1		5.26
Upsilon	υ	$\underline{b}$ b	0	1	9.46
Phi-meson	Φ	s $\underline{s}$	0	1	1.02
F-meson	F^{+}	c $\underline{s}$	0	1	2.04

observed by scattering experiments. The scattering appears to come from particles with charges of +2/3 and −1/3 of the electronic charge. (Recall that the up quark has a charge of +2/3, whereas the down quark has a charge of −1/3.) *There is thus, experimental evidence for the quark structure of the proton.* Similar experiments have also been performed on neutrons with the same success. The scattering also confirmed the existence of some quark-antiquark pairs within the proton. Recall that quark-antiquark pairs are the constituents of mesons. The experiments also showed the existence of other particles within the nucleons, called *gluons*. The gluons are the exchange particles between the quarks that act to hold the quarks together. They are the nuclear glue.

The one difficulty with the quark model at this point is that there seems to be a violation of the Pauli exclusion principle. Recall that the Pauli exclusion principle stated that no two electrons can have the same quantum numbers at the same time. The Pauli exclusion principle is actually more general than that, in that it applies not only to electrons, but to any particles that have half-integral spin, such as 1/2, 3/2, 5/2, and so on. *Particles that have half-integral spin are called fermions.* Because quarks have spin 1/2, they also must obey the Pauli exclusion principle. But the Δ^{++} particle is composed of three up quarks all with the same spin, and the Ω^- particle has three strange quarks all with the same spin. *Thus, there must be an additional characteristic of each quark, that is different for each quark, so that the Pauli exclusion principle will not be violated. This new attribute of the quark is called "color."*

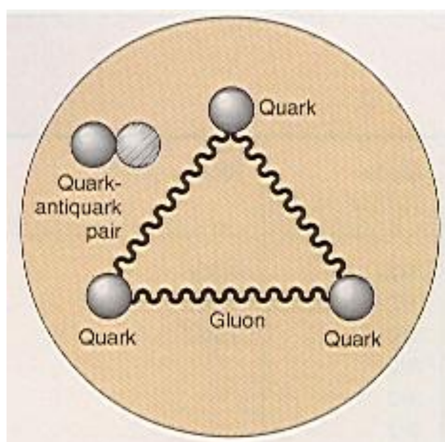


Figure 6.5 Structure of the proton. (From D. H. Perkins, "Inside the Proton" in The Nature of Matter, Clarendon Press, Oxford. 1981)

Quarks come in three colors: red, green, and blue. We should note that these colors are just names and have no relation to the real colors that we see everyday with our eyes. The words are arbitrary. As an example, they could just as easily have been called A, B, and C. We can think of color in the same way as electric charges. Electric charges come in two varieties, positive and negative. Color charges come in three varieties: red, green, and blue. *Thus, there are three types of up* *quarks; a red-up quark u_R, a green-up quark u_G, and a blue-up quark u_B.* Hence the delta plus-plus particle Δ^{++} can be represented as in figure 6.6(a). In this way there is no violation of the Pauli exclusion principle since each up quark is different.

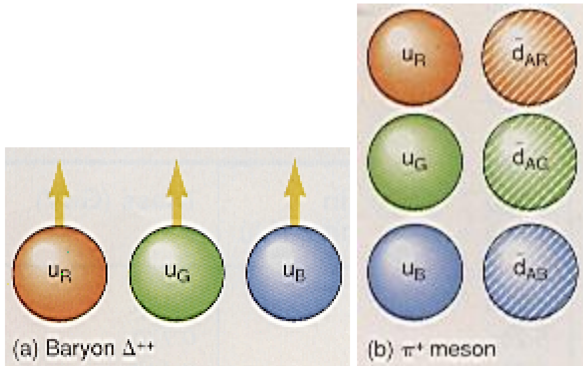
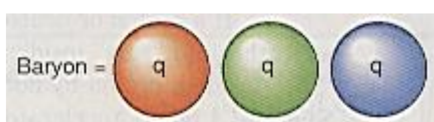
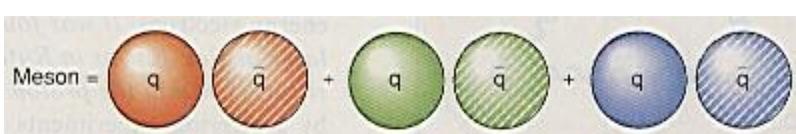


Figure 6.6 Colored quarks.

All baryons are composed of red, green, and blue quarks. Just as the primary colors red, green, and blue add up to white, the combination of a red, green, and blue quark is said to make up the color white. All baryons are, therefore, said to be white, or colorless. Just as a quark has an antiquark, each color of quark has an anticolor. Hence, a red-up quark has an up antiquark that carries the color antired, and is called an antired-up quark. The varieties of quarks are called flavors, such as up, down, strange, and so on. Hence, each flavor of quark comes in three colors to give a total of six flavors times three colors equals 18 quarks. Associated with the 18 quarks are 18 antiquarks. *Mesons, like baryons, must also be white or colorless. Hence, one colored quark of a meson must always be associated with an anticolor, since a color plus its anticolor gives white.* Thus, possible formations of a π^+ meson are shown in figure 6.6(b). That is, a red-up quark u_R combines with an antidown quark that carries the color antired $\bar{d}_{AR}$ to form the white π^+ meson. (The anticolor quark is shown with the hatched lines in figure 6.6.) Similarly the π^+ meson can be made out of green and antigreen $u_G\bar{d}_{AG}$ and blue and antiblue quarks $u_B\bar{d}_{AB}$ and a linear combination of them, such as $u_R\bar{d}_{AR} + u_G\bar{d}_{AG} + u_B\bar{d}_{AB}$ We can rewrite equations 6.4 and 6.5 as



(6.6)



(6.7)

The force between a quark carrying a color and its antiquark carrying anticolor is always attractive. Similarly the force between three quarks each of a different color is also attractive. All other combinations of colors gives a repulsive force. We will say more about colored quarks when we discuss the strong nuclear force in section 6.8.

6.5 The Electromagnetic Force

The electromagnetic force has been discussed in some detail in your previous general physics course. To summarize the results from there, Coulomb's law gave the electric force between charged particles, and the electric field was the mediator of that force. The relation between electricity and magnetism was first discovered by Ampère when he found that a current flowing in a wire produced a magnetic field. Faraday found that a changing magnetic field caused an electric current. James Clerk Maxwell synthesized all of electricity with all of magnetism into his famous equations of electromagnetism. That is, the separate force of electricity and the force of magnetism were unified into one electromagnetic force.

The merger of electromagnetic theory with quantum mechanics has led to what is now called ***quantum electrodynamics,*** *which is abbreviated* ***QED****. In QED the electric force is transmitted by the exchange of a virtual photon.* That is, the force between two electrons can be visualized as in figure 6.7. Recall from chapter 3 that the Heisenberg uncertainty relation allows for the creation of a virtual particle as long as the energy associated with the mass of the virtual particle is repaid in a time interval Δt that satisfies equation 3.56. In figure 6.7, two electrons approach each other. The first electron emits a virtual photon and recoils as shown.

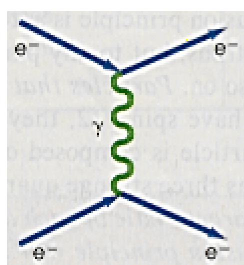


Figure 6.7 The electric force as an exchange of a virtual photon.

When the second electron absorbs that photon it also recoils as shown, leading to the result that the exchange of the photon caused a force of repulsion between the two electrons. As pointed out in chapter 3, this exchange force is strictly a quantum mechanical phenomenon with no real classical analogue. So it is perhaps a little more difficult to visualize that the exchange of a photon between an electron and a proton produces an attractive force between them. *The exchanged photon is the mediator or transmitter of the force. All of the forces of nature can be represented by an exchanged particle.*

Because the rest mass of a photon is equal to zero, the range of the electric force is infinite. This can be shown with the help of a few equations from chapter 3. The payback time for the uncertainty principle was

$$\Delta t = \frac{\hbar}{\Delta E} \tag{3.56}$$

While the energy ΔE was related to the mass Δm of the virtual particle by

$$\Delta E = (\Delta m)c^2$$

Substituting this into equation 3.56, gave for the payback time

$$\Delta t = \frac{\hbar}{(\Delta m)c^2} \tag{3.57}$$

The distance a virtual particle could move and still return during that time Δt, was given as

$$d = c\,\frac{\Delta t}{2} \tag{3.58}$$

This distance is called the *range* of the virtual particle. Substituting equation 3.57 into 3.58 gives for the range

$$d = \frac{c\,\hbar}{2(\Delta m)c^2}$$

$$d = \frac{\hbar}{2c}\,\frac{1}{\Delta m} \tag{6.8}$$

For a photon, the rest mass Δm is equal to zero. So as the denominator of a fraction approaches zero, the fraction approaches infinity. Hence, the range d of the particle goes to infinity. Thus, the electric force should extend to infinity, which, of course, it does.

6.6 The Weak Nuclear Force

The weak nuclear force is best known for the part it plays in radioactive decay. Recall from chapter 5 on nuclear physics that the initial step in beta β^- decay is for a neutron in the nucleus to decay according to the relation

$$\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \tag{6.9}$$

Whereas the proton inside the nucleus decays as

$$p \rightarrow n + e^{+} + \nu_e \tag{6.10}$$

and is the initial step in the beta β^{+} decay. Finally, the radioactive disintegration caused by the capture of an electron by the nucleus (electron capture), is initiated by the reaction

$$e^{-} + p \rightarrow n + \nu_e \tag{6.11}$$

These three reactions are just some of the reactions that are mediated by the weak nuclear force.

*The **weak nuclear force** does not exert the traditional push or pull type of force known in classical physics. Rather, it is responsible for the transmutation of the subatomic particles. The weak nuclear force is independent of electric charge and acts between leptons and hadrons and also between hadrons and hadrons.* The range of the weak nuclear force is very small, only about 10^{-17} m. The decay time is relatively large in that the weak decay occurs in about 10^{-10} seconds, whereas decays associated with the strong interaction occur in approximately 10^{-23} seconds.

The weak nuclear force is the weakest force after gravity. A product of weak interactions is the neutrino. The neutrinos are very light particles. Some say they have zero rest mass while others consider them to be very small, with an upper limit of about 10^{-30} eV for the ν_e neutrino. The neutrino is not affected by the strong or electromagnetic forces, only by the weak force. Its interaction is so weak that it can pass through the earth or the sun without ever interacting with anything.

6.7 The Electroweak Force

Steven Weinberg, Abdus Salam, and Sheldon Glashow proposed a unification of the electromagnetic force with the weak nuclear force and received the Nobel Prize for their work in 1979. This force is called the **electroweak force.** Just as a virtual photon mediates the electromagnetic force between charged particles, it became obvious that there should also be some particle to mediate the weak nuclear force. The new electroweak force is mediated by four particles: the photon and three intermediate *vector bosons* called W^{+}, W^{-}, and Z^{0}. The photon mediates the electromagnetic force, whereas the vector bosons mediate the weak nuclear force. In terms of the exchange particles, the decay of a neutron, equation 6.9, is shown in figure 6.8(a). *A neutron decays by emitting a W^{-} particle, thereby converting the neutron into a proton. The W^{-} particle subsequently decays within 10^{-26} s into an electron and an antineutrino.* The decay of the proton in a radioactive nucleus, equation 6.10, is shown in figure 6.8(b). *The proton emits the positive intermediate vector boson, W^{+}, and is converted into a neutron. The W^{+} subsequently decays into a positron and a neutrino.* An electron capture, equation 6.11, is shown in figure 6.8(c) as a collision between a proton and an electron. The proton emits a W^{+} and is converted into a neutron. The W^{+} then combines with the electron forming a neutrino. The Z^{0} particle is observed in electron-neutrino scattering, as shown in figure 6.8(d).

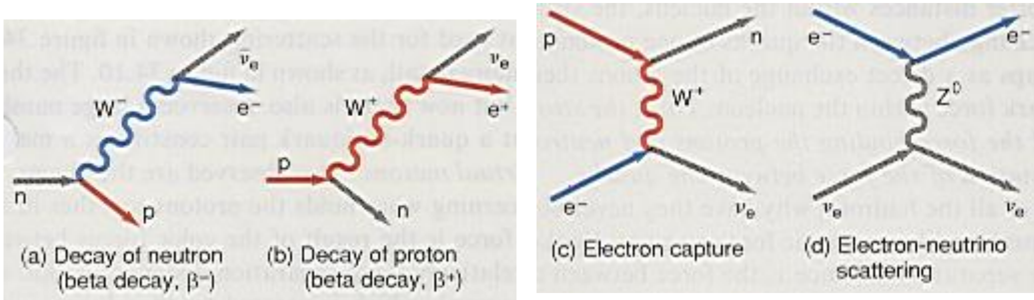


Figure 6.8 Examples of the electroweak force.

The vector bosons, W^+ and W^-, were found experimentally in proton-antiproton collisions at high energies, at the European Center for Nuclear Research (CERN), in January 1983, by a team headed by Carlo Rubbia of Harvard University. The Z^0 was found a little later in May 1983. The mass of the $W^{\pm}$ was around 80 GeV, while the mass of the Z^0 was about 90 GeV. Referring to equation 6.8, we see that for such a large mass, Δm in that equation gives a very short range d for the weak force, as found experimentally.

At very high energies, around 100 GeV, the electromagnetic force and the weak nuclear force merge into one electroweak force that acts equally between all particles: hadrons and leptons, charged and uncharged.

6.8 The Strong Nuclear Force

As mentioned previously, the **strong nuclear force** is responsible for holding the protons and neutrons together in the nucleus. The strong nuclear force must indeed be very strong to overcome the enormous electrical force of repulsion between the protons. Yukawa proposed that an exchange of mesons between the nucleons was the source of the nuclear force. But the nucleons are themselves made up of quarks. What holds these quarks together?

*In quantum electrodynamics (QED), the electric force was caused by the exchange of virtual photons. One of the latest theories in elementary particle physics is called **quantum chromodynamics (QCD)** and the force holding quarks together is caused by the exchange of a new particle, called a "gluon."* That is, a gluon is the nuclear glue that holds quarks together in a nucleon. Figure 6.9(a) shows the force between quarks as the exchange of a virtual gluon. Gluons, like quarks, come in colors and anticolors. *A gluon interacting with a quark changes the color of a quark.* As an example, figure 6.9(b) shows a red-up quark u_R emitting a red-antiblue gluon $(R\overline{B})$ The up quark loses its red color and becomes blue. That is, *in taking away an anticolor, the color itself must remain.* Hence, taking away an antiblue from the up quark, the color blue must remain. When the first blue- up quark

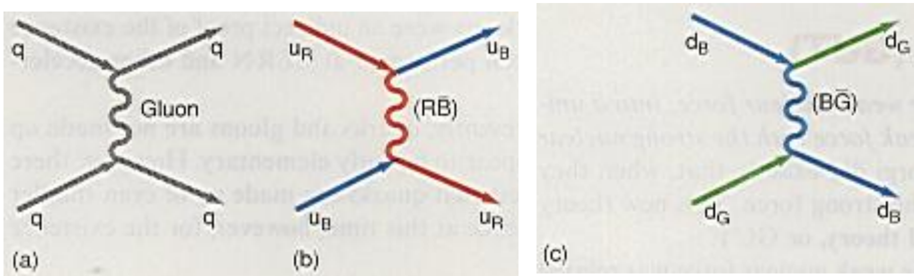


Figure 6.9 Exchange of gluons between quarks.

receives the red-antiblue gluon $(R\overline{B})$, the blue of the up quark combines with the antiblue of the gluon canceling out the color blue. (A color and its anticolor always gives white.) The red color of the gluon is now absorbed by the up quark turning it into a red-up quark. *Thus, in the process of exchanging the gluon, the quarks changed color*. Figure 6.9(c) shows a blue-down quark emitting a blue-antigreen gluon $(B\overline{G})$, changing the blue-down quark into a green-down quark. When the first green-down quark absorbs the $(B\overline{G})$ gluon, the color green cancels and the down quark becomes a blue-down quark.

All told, there are eight different gluons and each gluon has a mass. Each gluon always carries one color and one anticolor. Occasionally a gluon can transform to a quark-antiquark pair.

At energies greater than that used for the scattering shown in figure 6.5, scattering from protons reveals even more detail, as shown in figure 6.10. The three valence quarks are shown as before, but now there is also observed a large

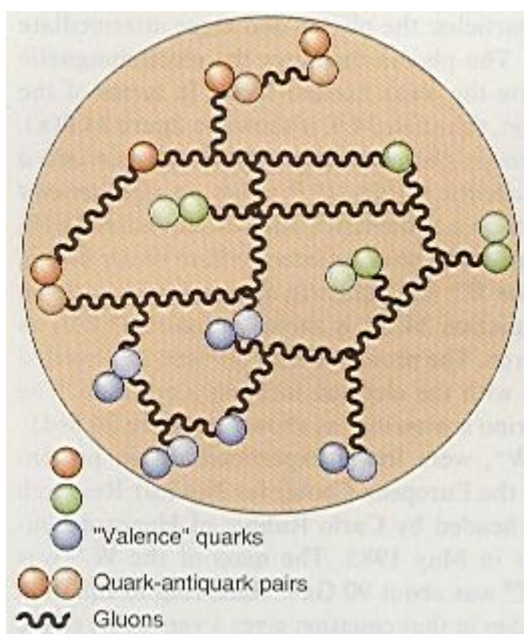


Figure 6.10 More detailed structure of the proton. (After D. H. Perkins, "The Nature of Matter", Oxford University Press)

number of quark-antiquark pairs. Recall that a quark-antiquark pair constitutes a meson. *Hence, the proton is seething with virtual mesons.* Also observed are the gluons. To answer the traditional questions concerning what holds the protons together in the nucleus, we can say that the strong force is the result of the color forces between the quarks within the nucleons. At relatively large separation distances within the nucleus, the quark-antiquark pair (meson), which is created by the gluons, is exchanged between the nucleons. At shorter distances within the nucleus, the strong force can be explained either as an exchange between the quarks of one proton and the quarks of another proton, or perhaps as a direct exchange of the gluons themselves, which give rise to the quark-quark force within the nucleon. *Thus, the strong force originates with the quarks, and the force binding the protons and neutrons together in the nucleus is the manifestation of the force between the quarks.*

If quarks are the constituents of all the hadrons, why have they never been isolated? The quark-quark force is something like an elastic force given by Hooke's law, $F = kx$. For small values of the separation distance x, the force between the quarks is small and the quarks are relatively free to move around within the particle. However, if we try to separate the quarks through a large separation distance x, then the force becomes very large, so large, in fact, that the quarks cannot be separated at all. *This condition is called the confinement of quarks*. *Thus, quarks are never seen in an isolated state because they cannot escape from the particle in which they are constituents.*

But is there any evidence for the existence of quarks? The answer is yes. Experiments were performed in the new PETRA storage ring at DESY (Deutsches Electronen-Synchronton) in Hamburg, Germany, in 1978. Electrons and positrons, each at an energy of 20 GeV, were fired at each other in a head-on collision. The annihilation of the electron and its antiparticle, the positron, produce a large amount of energy; it is from this energy that the quarks are produced. The experimenters found a series of "quark jets," which were the decay products of the quarks, exactly as predicted. (A quark jet is a number of hadrons flying off from the interaction in roughly the same direction.) These quark jets were an indirect proof of the existence of quarks. Similar experiments have been performed at CERN and other accelerators.

As far as can be determined presently, quarks and gluons are not made up of still smaller particles; that is, they appear to be truly elementary. However, there are some speculative theories that suggest that quarks are made up of even smaller particles called preons. There is no evidence at this time, however, for the existence of preons.

6.9 Grand Unified Theories (GUT)

If it is possible to merge the electric force with the weak nuclear force, into a unified electroweak force, why not merge the electroweak force with the strong nuclear force?

In 1973 Sheldon Glashow and Howard Georgi did exactly that, when they published a theory merging the electroweak with the strong force. This new theory was the first of many to be called the **grand unified theory,** or GUT.

The first part of this merger showed how the weak nuclear force was related to the strong nuclear force. Let us consider the decay of the neutron shown in equation 6.9:

$$n \rightarrow p + e^- + \bar{\nu}_e$$

We can now visualize this decay according to the diagram in figure 6.11. According

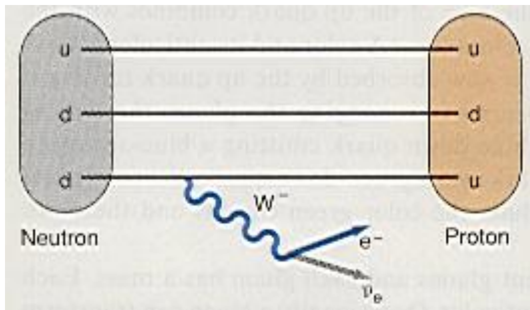


Figure 6.11 The decay of the neutron.

to the quark theory, a neutron is composed of one up quark and two down quarks. One of the down quarks of the neutron emits the W^- boson and is changed into an up quark, transforming the neutron into a proton. (Recall, that the proton consists of two up quarks and one down quark.) The W^- boson then decays into an electron and an antineutrino. Thus, *the weak force changes the flavor of a quark, whereas the strong force changes only the color of a quark.*

Above 10^{15} GeV of energy, called the *grand unification energy,* we can no longer tell the difference between the strong, weak, and electromagnetic forces. Above this energy there is only one unified interaction or force that occurs. Of course, this energy is so large that it is greater than anything we could ever hope to create experimentally. As we shall see, however, it could have been attained in the early stages of the creation of the universe — the so-called “Big Bang.”

The strong nuclear force operates between quarks, whereas the weak nuclear force operates between quarks and leptons. If the strong and weak forces are to be combined, then the quarks and leptons should be aspects of one more fundamental quantity. That is, the grand unified force should be able to transform quarks into leptons and vice versa. In the grand unified theories, there are 24 particles that mediate the unified force and they are listed in table 6.5. In grand unified theories, the forces are unified because the forces arise through the exchange of the same family of particles. As seen before, the photon mediates the electromagnetic force; the vector bosons mediate the weak force; the gluons mediate the strong force; and there are now 12 new particles called X particles (sometimes progenitor and/or lepto-quark particles) that mediate the unified force. It is these X particles that are capable of converting hadrons into leptons by changing quarks to leptons.

Table 6.5 Family of Particles that Mediate the Unified Force		
Particle	Number	Force Mediated
Photon	1	Electromagnetic
Vector bosons (W^+, W^-, Z^0)	3	Weak
Gluons	8	Strong
X particles	12	Strong-electroweak

The X particles come in four different electrical charges, ±1/3 and ±4/3. Thus, the X particles can be written as $X^{1/3}$, $X^{-1/3}$, $X^{4/3}$, and $X^{-4/3}$. Each of these X particles also comes in the three colors red, blue, and green, thereby giving the total of 12 X particles. The X particles can change a quark into a lepton, as shown in figure 6.12. An X particle carrying an electrical charge of −4/3, and a color charge

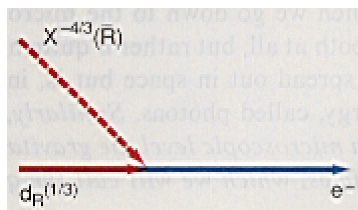


Figure 6.12 Changing a quark to an electron.

of antired combines with a red-down quark, which carries an electrical charge of 1/3. The colors red and antired cancel to give white, while the electrical charge becomes 1/3 − 4/3 = −3/3 = −1 and an electron is created out of a quark. This type of process is not readily seen in our everyday life because the mass of the virtual X particle must be of the order of 10^{15} GeV, which is an extremely large energy. A similar analysis shows that an isolated proton should also decay. The lifetime, however, is predicted to be 10^{32} yr. Experiments are being performed to look for the predicted decays. However, at the present time no such decay of an isolated proton has been found. An isolated proton seems to be a very stable particle, indicating that either more experiments are needed, or the GUT model needs some modifications.

6.10 The Gravitational Force and Quantum Gravity

As has been seen throughout this book, physics is a science of successive approximations to the truth hidden in nature. Newton found that celestial gravity was of the same form as terrestrial gravity and unified them into his law of universal gravitation. However, it turned out that it was not quite so universal. Einstein started the change in his special theory of relativity, which governed systems moving with respect to each other at constant velocity. As he generalized this theory to systems that were accelerated with respect to each other, he found the equivalence between accelerated systems and gravity. The next step of course was to show that matter warped spacetime and gravitation was a manifestation of that warped spacetime. Thus, general relativity became a law of gravitation, and it was found that Newton's law of gravitation was only a special case of Einstein's theory of general relativity.

We have also seen that the quantum theory is one of the great new theories of modern physics, which seems to say that nature is quantized. There are quanta of energy, mass, angular momentum, charge, and the like. But general relativity, in its present format, is essentially independent of the quantum theory. It is, in this sense, still classical physics. It, too, must be only an approximation to the truth hidden in nature. *A more general theory should fuse quantum mechanics with general relativity — that is, we need a quantum theory of gravity*.

In order to combine quantum theory with general relativity (hereafter called Einstein's theory of gravitation), we have to determine where these two theories merge. Remember the quantum theory deals with very small quantities, because of the smallness of Planck's constant h, whereas Einstein's theory of gravitation deals with very large scale phenomena, or at least with very large masses that can significantly warp spacetime.

One of the important characteristics of the quantum theory is the wave-particle duality; waves can act as particles and particles can act as waves. And as has also been seen, waves can exist in the electromagnetic field. Let us, for the moment, compare electromagnetic fields with gravitational fields. On a large scale the electric field appears smooth. It is only when we go down to the microscopic level that we see that the electric field is not smooth at all, but rather is quite bumpy, because the energy of the electric field is not spread out in space but is, instead, stored in little bundles of electromagnetic energy, called photons. *Similarly, from the quantum theory we should expect that on a microscopic level the gravitational field should also be quantized into little particles, which we will call the quanta of the gravitational field — the* ***gravitons.***

But what is a gravitational field but the warping of spacetime? Hence, a quantum of gravitation must be a quantum of spacetime itself. Thus, the graviton would appear to be a quantum of spacetime. Therefore, on a microscopic level, spacetime itself is probably not smooth but probably has a graininess or bumpiness to it. At this time, no one knows for sure what happens to spacetime on this microscopic level, but it has been conjectured that spacetime may look something like a foam that contains "wormholes."

At what point do the quantum theory and Einstein's theory of gravitation merge? The answer is to be found in Heisenberg's uncertainty principle.

$$\Delta E \Delta t \geq \hbar \tag{31.55}$$

For the electric field, small quantities of energy ΔE of the electric field are turned into small quanta of energy, the photons. In a similar manner, small quantities of energy ΔE of the gravitational field should be turned into little bundles or quantums of gravity, the gravitons. Since the range of a force is determined by the mass of the exchanged particle, and the range of the gravitational force is known to be infinite, it follows that the rest mass of the graviton must be zero. Hence, a quantum fluctuation should appear as a gravitational wave moving at the speed of light *c*. Therefore, if we consider a fluctuation of the gravitational field that spreads out spherically, the small time for it to move a distance *r* is

$$\Delta t = \frac{r}{c} \tag{6.12}$$

To obtain an order of magnitude for the energy, we drop the greater than sign in the uncertainty principle and on substituting equation 6.12 into equation 3.55 we get, for the energy of the fluctuation,

$$\Delta E \Delta t = \Delta E \frac{r}{c} = \hbar$$

and

$$\Delta E = \frac{\hbar c}{r} \tag{6.13}$$

The value of *r* in equation 6.13, wherein the quantum effects become important, is unknown at this point; in fact, it is one of the things that we wish to find. So further information is needed. Let us consider the amount of energy required to pull this little graviton or bundle of energy apart against its own gravity. The work to pull the graviton apart is equal to the energy necessary to assemble that mass by bringing small portions of it together from infinity. Let us first consider the problem for the electric field, and then use the analogy for the gravitational field. Recall that the electric potential for a small spherical charge is

$$V = \frac{kq}{r}$$

But the electric potential *V* was defined as the potential energy per unit charge, that is,

$$V = \frac{\text{PE}}{q}$$

So if a second charge *q* is brought from infinity to the position *r*, the potential energy of the system of two charges is

$$\text{PE} = qV = \frac{kq^2}{r}$$

In a similar vein, a gravitational potential Φ could have been derived using the same general technique used to derive the electric potential. The result for the gravitational potential would be

$$\Phi = \frac{GM}{r} \tag{6.14}$$

where *G,* of course, is the gravitational constant, *M* is the mass, and *r* is the distance from the mass to the point where we wish to determine the gravitational potential. The gravitational potential of a spherical mass is defined, similar to the electric potential, as the gravitational potential energy per unit mass. That is,

$$\Phi = \frac{\text{PE}}{M} \tag{6.15}$$

Hence, if another mass *M* is brought from infinity to the position *r,* the potential energy of the system of two equal masses is

$$\text{PE} = M\Phi = \frac{GM^2}{r} \tag{6.16}$$

This value of the potential energy, PE to assemble the two masses, is the same energy that would be necessary to pull the two masses apart. Applying the same reasoning to the assembly of the masses that constitutes the graviton, the potential energy given by equation 6.16 is equal to the energy that would be necessary to pull the graviton apart. This energy can be equated to the energy of the graviton found from the uncertainty principle. Thus,

$$\text{PE} = \Delta E$$

Substituting for the PE from equation 6.16 and the energy ΔE from the uncertainty principle, equation 6.13, we get

$$\frac{GM^2}{r} = \frac{\hbar c}{r} \tag{6.17}$$

But the mass of the graviton *M* can be related to the energy of the graviton by Einstein's mass-energy relation as

$$\Delta E = Mc^2$$

or

$$M = \frac{\Delta E}{c^2} \tag{6.18}$$

Substituting equation 6.18 into equation 6.17 gives

$$\frac{G(\Delta E)^2}{r(c^2)^2} = \frac{\hbar c}{r}$$

Solving for ΔE, we get

$$\Delta E = \sqrt{\frac{\hbar c^5}{G}} \tag{6.19}$$

Equation 6.19 represents the energy of the graviton.

Example 6.1

The energy of the graviton. Find the energy of the graviton.

Solution

The energy of the graviton, found from equation 6.19, is

$$\Delta E = \sqrt{\frac{\hbar c^5}{G}}$$

$$\Delta E = \sqrt{\frac{(1.05\times10^{-34}\text{ J s})(3.00\times10^{8}\text{ m/s})^5}{6.67\times10^{-11}\text{ (N m}^2\text{)/kg}^2}}$$

$$\Delta E = 1.96\times10^{9}\text{ J}$$

This can also be expressed in terms of electron volts as

$$\Delta E = (1.96\times10^{9}\text{ J})\left(\frac{1\text{ eV}}{1.60\times10^{-19}\text{J}}\right)\left(\frac{1\text{ GeV}}{10^{9}\text{ eV}}\right)$$

$$= 1.20\times10^{19}\text{ GeV}$$

This is the energy of a graviton; it is called the *Planck energy*.

From the point of view of particle physics, then, the graviton looks like a particle of mass 10^{19} GeV/c^2. This is an enormous mass and energy when compared to the masses and energies of all the other elementary particles. However, for any elementary particles of this size or larger, both quantum theory and gravitation must be taken into account. Recall that in all the other interactions of the elementary particles, gravity was ignored. From the point of view of ordinary gravity, this energy is associated with a mass of 2×10^{-5} g, a very small mass.

The distance in which this quantum fluctuation occurs can now be found by equating ΔE from equation 6.13 to ΔE from equation 6.19, that is,

$$\Delta E = \frac{\hbar c}{r} = \Delta E = \sqrt{\frac{\hbar c^5}{G}}$$

Solving for r we get

$$r = \frac{\hbar c}{\sqrt{\hbar c^5/G}}$$

$$r = \sqrt{\frac{\hbar G}{c^3}} \tag{6.20}$$

Equation 6.20 is the distance or length where quantum gravity becomes significant. This distance turns out to be the same distance that Max Planck found when he was trying to establish some fundamental units from the fundamental constants of nature, and is called the *Planck length* L_P. Hence, the Planck length is

$$L_P = \sqrt{\frac{\hbar G}{c^3}} \tag{6.21}$$

Example 6.2

The Planck length. Determine the size of the Planck length.

Solution

The Planck length, determined from equation 6.21, is

$$L_P = \sqrt{\frac{\hbar G}{c^3}}$$

$$L_P = \sqrt{\frac{(1.05 \times 10^{-34}\text{J s})(6.67 \times 10^{-11}(\text{N m}^2)/\text{kg}^2)}{(3.00 \times 10^8 \text{m/s})^3}}$$

$$= 1.61 \times 10^{-35}\text{ m} = 1.61 \times 10^{-33}\text{cm}$$

Thus, quantum fluctuations of spacetime start to occur at distances of the order of 1.61×10^{-33} cm. We can now find the interval of time, within which this quantum fluctuation of spacetime occurs, from equation 6.12 as

$$\Delta t = \frac{r}{c} = \frac{L_P}{c}$$

This time unit is called the *Planck time* T_P and is

$$T_P = \frac{L_P}{c} \tag{6.22}$$

$$= \frac{1.61 \times 10^{-35}\text{m}}{3.00 \times 10^8 \text{m/s}}$$

$$= 5.37 \times 10^{-44}\text{s}$$

Thus, intervals of space and time given by the Planck length and the Planck time are the regions in which quantum gravity must be considered. This distance and time are extremely small. Recall that the size of the electron is about 10^{-19} m. Thus, quantum gravity occurs on a scale much smaller than that of an atom, a nucleus, or even an electron. There is relatively little known about quantum gravity at this time, but research is underway to find more answers dealing with the ultimate structure of spacetime itself.

6.11 The Superforce — Unification of All the Forces

An attempt to unify all the forces into one single force — a kind of **superforce** — continues today. One of the techniques followed is called *supersymmetry,* where the main symmetry element is spin. (Recall that all particles have spin.) Those particles that obey the Pauli exclusion principle have half-integral spin, that is, spin $\hbar/2$, $3\hbar/2$, and so on. Those particles that obey the Pauli exclusion principle are called *fermions.* All the quarks and leptons are fermions. Particles that have integral spin, $\hbar$, $2\hbar$, and so on, do not obey the Pauli exclusion principle. These particles are called *bosons.* All the mediating particles, such as the photon, $W^{\pm}$, Z^0, gluons, and the like, are bosons. *Hence, fermions are associated with particles of matter, whereas bosons are associated with the forces of nature, through an exchange of bosons. The new theories of supersymmetry attempt to unite bosons and fermions.*

A further addition to supersymmetry unites gravity with the electroweakstrong or GUT force into the superforce that is also called super gravity. Super gravity requires not only the existence of the graviton but also a new particle, the “gravitino,” which has spin 3/2. However, this unification exists only at the extremely high energy of 10^{19} GeV, an energy that cannot be produced in a laboratory. However, in the initial formation or creation of the universe, a theory referred to as the Big Bang, such energies did exist.

The latest attempt to unify all the forces is found in the *superstring theory.* The superstring theory assumes that the ultimate building blocks of nature consist of very small vibrating strings. As we saw in our study of wave motion, a string is capable of vibrating in several different modes. The superstring theory assumes that each mode of vibration of a superstring can represent a particle or a force. Because there are an infinite number of possible modes of vibration, the superstring can represent an infinite number of possible particles. The graviton, which is responsible for the gravitational interaction, is caused by the lowest vibratory mode of a circular string. (Superstrings come in two types: open strings, which have ends, and closed strings, which are circular.) The photon corresponds to the lowest mode of vibration of the open string. Higher modes of vibrations represent different particles, such as quarks, gluons, protons, neutrons, and the like. In fact, the gluon is considered to be a string that is connected to a quark at each end. In this theory, no particle is more fundamental than any other, each is just a different mode of vibration of the superstrings. The superstrings interact with other superstrings by breaking and reforming. The four forces are considered just different manifestations of the one unifying force of the superstring. The superstring theory assumes that the universe originally existed in ten dimensions, but broke into two pieces — one of the pieces being our four-dimensional universe. Like the theories of supersymmetry and super gravity, the energies needed to test this theory experimentally are too large to be produced in any laboratory.

A simple picture of the unifications is shown in table 6.6. A great deal more work is necessary to complete this final unification.

Table 6.6
The Forces and Their Unification

Electricity				
	Electromagnetism			
Magnetism		Electroweak force		
Weak force			Grand unified theories (GUT)	
Strong force				Superforce
Gravity				

Have you ever wondered … ?
An Essay on the Application of Physics
The Big Bang Theory and the Creation of the Universe

Have you ever wondered how the world was created? In every civilization throughout time and throughout the world, there has always been an account of the creation of the world. Such discussions have always belonged to religion and philosophy. It might seem strange that astronomers, astrophysicists, and physicists have now become involved in the discussion of the creation of the universe. Of course, if we think about it, it is not strange at all. *Since physics is a study of the entire physical world; it is only natural that physics should try to say something about the world's birth.*

The story starts in 1923 when the American astronomer, Edwin Hubble, using the Doppler effect for light, observed that all the galactic clusters, outside our own, in the sky were receding away from the earth. When we studied the Doppler effect for sound, we saw that when a train recedes from us its frequency decreases. A decrease in the frequency means that there is an increase in the wavelength. Similarly, a Doppler effect for light waves can be derived. The equations are different than those derived for sound because, in the special theory of relativity, the velocity of light is independent of the source. However, the effect is the same. That is, a receding source that emits light at a frequency ν, is observed by the stationary observer to have a frequency ν', where ν' is less than ν. Thus, since the frequency decreases, the wavelength increases. Because long waves are associated with the red end of the visible spectrum, all the observed wavelengths are shifted toward the red end of the spectrum. The effect is called the *cosmological red shift,* to distinguish it from the gravitational red shift discussed in chapter 2. *Hubble found that the light from the distant galaxies were all red shifted indicating that the distant galaxies were receding from us.*

It can, therefore, be concluded that if all the galaxies are receding from us, the universe itself must be expanding. Hubble was able to determine the rate at which the universe is expanding. If the universe is expanding now, then in some time in the past it must have been closer together. If we look far enough back in time, we should be able to find when the expansion began. (Imagine taking a movie picture of an explosion showing all the fragments flying out from the position of the explosion. If the movie is run backward, all the fragments would be seen moving backward toward the source of the explosion.)

The best estimate for the creation of the universe, is that the universe began as a great bundle of energy that exploded outward about 15 billion years ago. This great explosion has been called the ***Big Bang.*** *It was not an explosion of matter into an already existing space and time, rather it was the very creation of space and time, or spacetime, and matter themselves.*

As the universe expanded from this explosion, all objects became farther and farther apart. A good analogy to the expansion of spacetime is the expansion of a toy balloon. A rectangular coordinate system is drawn on an unstretched balloon, as shown in figure 1(a), locating three arbitrary points, *A, B,* and *C*. The balloon is

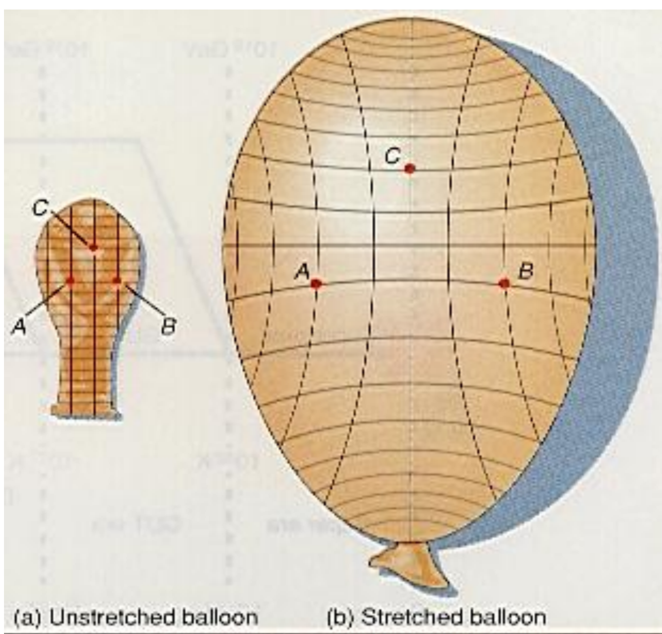


Figure 1 An Analogy to the expanding universe.

then blown up. As the balloon expands the distance between points *A* and *B, A* and *C,* and *B* and *C* increases. So no matter where you were on the surface of the balloon you would find all other points moving away from you. This is similar to the distant galaxies moving away from the earth. To complete the analogy to the expanding universe, we note that the simple flat rectangular grid in which Euclidean geometry holds now become a curved surface in which Euclidean geometry no longer holds.

If everything in the universe is spread out and expanding, the early stages of the universe must have been very compressed. To get all these masses of stars of the present universe back into a small compressed state, that compressed state must have been a state of tremendous energy and exceedingly high density and temperature. Matter and energy would be transforming back and forth through Einstein's mass-energy formula, $E = mc^2$. Work done by particle physicists at very high energies allows us to speculate what the universe must have looked like at these very high energies at the beginning of the universe.

The early history of the universe is sketched in figure 2. The Big Bang is shown occurring at time $t = 0$, which is approximately 15 billion years ago.

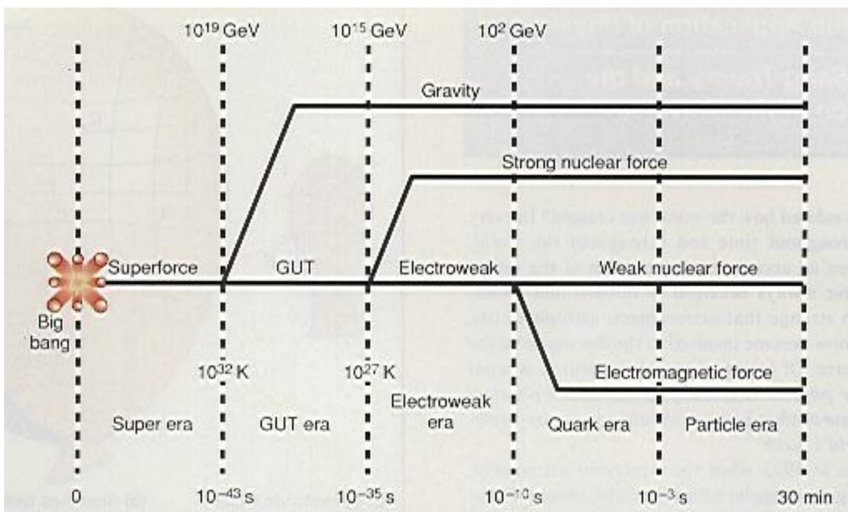


Figure 2 Creation of the four forces from the superforce.

1. *From the Big Bang to 10^{-43} s*

   Between the creation and the Planck time, 0 to 10^{-43} s, the energy of the universe was enormous, dropping to about 10^{19} GeV at the Planck time. The temperature was greater than 10^{33} K. Relatively little is known about this era, but the extremely high energy would cause all the forces to merge into one superforce. That is, gravity, the strong force, the weak force, and the electromagnetic force would all be replaced by one single superforce. This is the era being researched by present physicists in the supersymmetry and super gravity theories. There is only one particle, a super particle, that decays into bosons and fermions, and continually converts fermions to bosons and vice versa, so that there is no real distinction between them.

2. *From 10^{-43} s to 10^{-35} s*
As the universe expands, the temperature drops and the universe cools to about 10^{32} K. The energy drops below 10^{19} GeV and the gravitational force breaks away from the superforce as a separate force, leaving the grand unified force of GUT as a separate force. Now two forces exist in nature. We are now in the GUT era, that era governed by the grand unified theories. The X particle and its antiparticle $\overline{X}$ are in abundance. The X particles decay into quarks and leptons, whereas the $\overline{X}$ particles decay into antiquarks and antileptons. However, the decay rate of X and $\overline{X}$ are not the same and more particles than antiparticles are formed. This will eventually lead to the existence of more matter than antimatter in the universe. The X particles continually convert quarks into leptons and vice versa. There are plenty of quarks, electrons, neutrinos, photons, gluons, X particles, and their antiparticles present, but they have effectively lost their individuality.

3. *From 10^{-35} s to 10^{-10}* s
As further expansion of the universe continues, the temperature drops to 10^{27} K and the energy drops to 10^{15} GeV. At this low energy all the X particles disappear, and quarks and leptons start to have an individual identity of their own. No longer can they be converted into each other. The lower energy causes the strong nuclear force to break away leaving the electroweak force as the only unified force left. There are now three forces of nature: gravity, strong nuclear, and the electroweak. There are quarks, leptons, photons, neutrinos, $W^{\pm}$ and Z^0, and gluon particles present. It is still too hot for the quarks to combine.

4. *From 10^{-10} s to 10^{-3} s*
As the universe continues to expand, it cools down to an energy of 10^2 GeV. The $W^{\pm}$ and Z^0 particles disappear because there is not enough energy to form them anymore. The weak nuclear force breaks away from the electroweak force, leaving the electromagnetic force. There are now present the four familiar forces of nature: gravity, strong nuclear, weak nuclear, and electromagnetic. Quarks now combine to form baryons, qqq, and mesons, $q\overline{q}$ The familiar protons and neutrons are now formed. Because of the abundance of quarks over antiquarks, there will also be an excess of protons and neutrons over antiprotons and antineutrons.

5. *From 10^{-3} s to 30 mi*n
The universe has now expanded and cooled to the point where protons and neutrons can combine to form the nucleus of deuterium. The deuterium nuclei combine to form helium as described in section 5.9 on fusion. There are about 77% hydrogen nuclei, and 23% helium nuclei present at this time and this ratio will continue about the same to the present day. There are no atoms formed yet because the temperature is still too high. What is present is called a *plasma*.

6. *From 30 min to 1 Billion Years*
   Further expansion and cooling now allows the hydrogen and helium nuclei to capture electrons and the first chemical elements are born. Large clouds of hydrogen and helium are formed.

7. *From 1 Billion Years to 10 Billion Years*
   The large rotating clouds of hydrogen and helium matter begin to concentrate due to the gravitational force. As the radius of the cloud decreases, the angular velocity of the cloud increases in order to conserve angular momentum. (Similar to the spinning ice skater) These condensing, rotating masses are the beginning of galaxies.
   Within the galaxies, gravitation causes more and more matter to be compressed into spherical objects, the beginning of stars. More and more matter gets compressed until the increased pressure of that matter causes a high enough temperature to initiate the fusion process of converting hydrogen to helium and the first stars are formed. Through the fusion process, more and more chemical elements are formed. The higher chemical elements are formed by neutron absorption until all the chemical elements are formed.
   These first massive stars did not live very long and died in an explosion — a supernova — spewing the matter of all these heavier elements out into space. The fragments of these early stars would become the nuclei of new stars and planets.

8. *From 10 Billion Years to the Present*
   The remnants of dead stars along with hydrogen and helium gases again formed new clouds, which were again compressed by gravity until our own star, the sun, and the planets were formed. All the matter on earth is the left over ashes of those early stars. Thus, even we ourselves are made up of the ashes of these early stars. As somebody once said, there is a little bit of star dust in each of us.

The Language of Physics

Leptons
Particles that are not affected by the strong nuclear force (p.).

Hadrons
Particles that are affected by the strong nuclear force (p.).

Baryons
A group of hadrons that have half-integral spin and are composed of three quarks (p.).

Mesons
A group of hadrons that have integral spin, that are composed of quark-antiquark pairs (p.).

Antiparticles
To each elementary particle in nature there corresponds another particle that has the characteristics of the original particle but opposite charge. Some neutral particles have antiparticles that have opposite spin, whereas the photon is its own antiparticle. The antiparticle of the proton is the antiproton. The antiparticle of the electron is the antielectron or positron. If a particle collides with its antiparticle both are annihilated with the emission of radiation or other particles. Conversely, photons can be converted to particles and antiparticles (p.).

Antimatter
Matter consists of protons, neutrons, and electrons, whereas antimatter consists of antiprotons, antineutrons, and antielectrons (p.).

Quarks
Elementary particles that are the building blocks of matter. There are six quarks and six antiquarks. The six quarks are: up, down, strange, charmed, bottom, and top. Each quark and antiquark also comes in three colors, red, green, and blue. Each color quark also has an anticolor quark. Baryons are composed of red, green, and blue quarks and mesons are made up of a linear combination of colored quark-antiquark pairs (p.).

Quantum electrodynamics (QED)
The merger of electromagnetic theory with quantum mechanics. In QED, the electric force is transmitted by the exchange of a virtual photon (p.).

Weak nuclear force
The weak nuclear force does not exert the traditional push or pull type of force known in classical physics. Rather, it is responsible for the transmutation of the subatomic particles. The weak force is independent of electric charge and acts between leptons and hadrons and also between hadrons and hadrons. The weak force is the weakest force after gravity (p.).

Electroweak force
A unification of the electromagnetic force with the weak nuclear force. The force is mediated by four particles: the photon and three intermediate vector bosons called W^+, W^-, and Z^0 (p.).

The strong nuclear force
The force that holds the nucleons together in the nucleus. The force is the result of the color forces between the quarks within the nucleons. At relatively large

separation distances within the nucleus, the quark-antiquark pair (meson), which is created by the gluons, is exchanged between the nucleons. At shorter distances within the nucleus, the strong force can be explained either as an exchange between the quarks of one proton and the quarks of another proton, or perhaps as a direct exchange of the gluons themselves, which give rise to the quark-quark force within the nucleon (p.).

Quantum chromodynamics (QCD)
In QCD, the force holding quarks together is caused by the exchange of a new particle, called a gluon. A gluon interacting with a quark changes the color of a quark (p.).

Grand unified theory
A theory that merges the electroweak force with the strong nuclear force. This force should be able to transform quarks into leptons and vice versa. The theory predicts the existence of 12 new particles, called X particles that are capable of converting hadrons into leptons by changing quarks to leptons. This theory also predicts that an isolated proton should decay. However, no such decays have ever been found, so the theory may have to be modified (p.).

Gravitons
The quanta of the gravitational field. Since gravitation is a warping of spacetime, the graviton must be a quantum of spacetime (p.).

Superforce
An attempt to unify all the forces under a single force. The theories go under the names of supersymmetry, super gravity, and superstrings (p.).

The Big Bang theory
The theory of the creation of the universe that says that the universe began as a great bundle of energy that exploded outward about 15 billion years ago. It was not an explosion of matter into an already existing space and time, rather it was the very creation of spacetime and matter (p.).

Questions for Chapter 6

*1. Discuss the statement, “A graviton is a quantum of gravity. But gravity is a result of the warping of spacetime. Therefore, the graviton should be a quantum of spacetime. But just as a quantum of the electromagnetic field, the photon, has energy, the graviton should also have energy. In fact, we can estimate the energy of a graviton. Therefore, is spacetime another aspect of energy? Is there only one fundamental quantity, energy?”

*2. Does antimatter occur naturally in the universe? How could you detect it? Where might it be located?

3. When an electron and positron annihilate, why are there two photons formed instead of just one?

4. Murray Gell-Mann first introduced three quarks to simplify the number of truly elementary particles present in nature. Now there are six quarks and six antiquarks, and each can come in three colors and three anticolors. Are we losing some of the simplicity? Discuss.

5. Discuss the experimental evidence for the existence of structure within the proton and the neutron.

6. How did the Pauli exclusion principle necessitate the introduction of colors into the quark model?

*7. If the universe is expanding from the Big Bang, will the gravitational force of attraction of all the masses in the universe eventually cause a slowing of the expansion, a complete stop to the expansion, and finally a contraction of the entire universe?

*8. Just as there are electromagnetic waves associated with a disturbance in the electromagnetic field, should there be gravitational waves associated with a disturbance in a gravitational field? How might such gravitational waves be detected?

*9. Einstein's picture of gravitational attraction is a warping of spacetime by matter. This has been pictured as the rubber sheet analogy. What might antimatter do to spacetime? Would it warp spacetime in the same way or might it warp spacetime to cause a gravitational repulsion? Would this be antigravity? Would the antiparticle of the graviton then be an antigraviton? Instead of a black hole, would there be a white hill?

10. Discuss the similarities and differences between the photon and the neutrino.

Problems for Chapter 6

Section 6.2 Particles and Antiparticles

1. How much energy is released when an electron and a positron annihilate? What is the frequency and wavelength of the two photons that are created?

2. How much energy is released when a proton and antiproton annihilate?

3. How much energy is released if 1.00 kg of matter annihilates with 1.00 kg of antimatter? Find the wavelength and frequency of the resulting two photons.

4. A photon "disintegrates," creating an electron-positron pair. If the frequency of the photon is 5.00×10^{24} Hz, determine the linear momentum and the energy of each product particle.

Section 6.4 Quarks

5. If the three quarks shown in the diagram combine to form a baryon, find the charge and spin of the resulting particle.

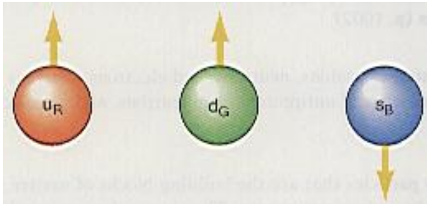


Diagram for problem 5.

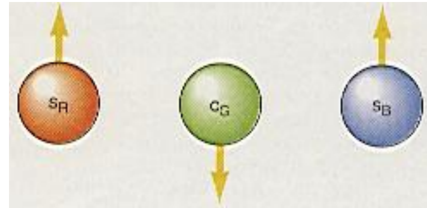


Diagram for problem 6.

6. If the three quarks shown in the diagram combine to form a baryon, find the charge and spin of the resulting particle.

7. If the three quarks shown in the diagram combine to form a baryon, find the charge and spin of the resulting particle.

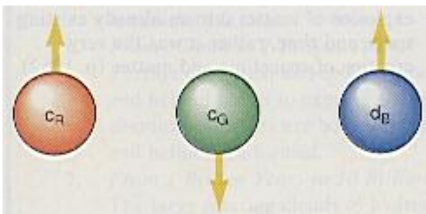


Diagram for problem 7.

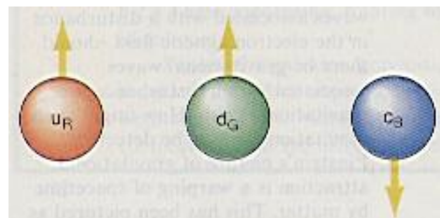


Diagram for problem 8.

8. Find the charge and spin of the baryon that consists of the three quarks shown in the diagram.

9. If the two quarks shown in the diagram combine to form a meson, find the charge and spin of the resulting particle.

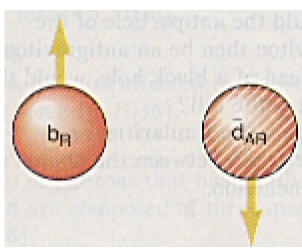


Diagram for problem 9.

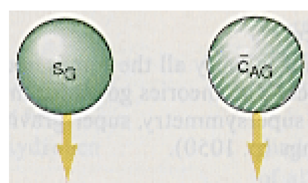


Diagram for problem 10.

10. If the two quarks shown in the diagram combine to form a meson, find the charge and spin of the resulting particle.

11. Find the charge and spin of the meson that consists of the two quarks shown in the diagram.

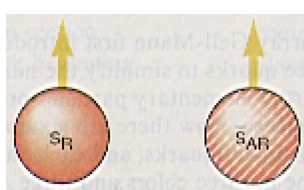


Diagram for problem 11.

12. Which of the combinations of particles in the diagram are possible and which are not. If the combination is not possible, state the reason.

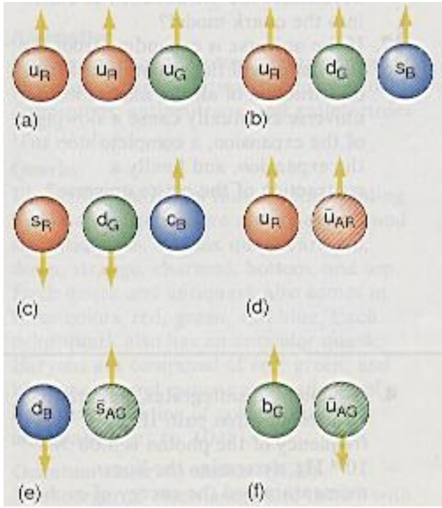


Diagram for problem 12.

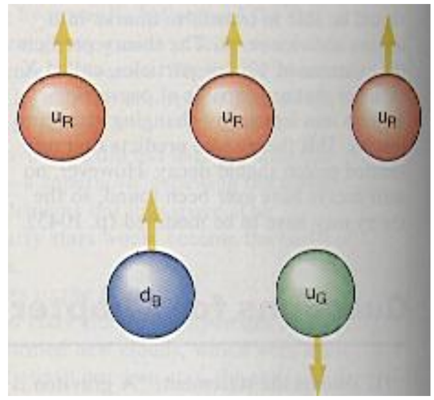


Diagram for problem 13.

13. Why are the two particles in the diagram impossible?

14. A baryon is composed of three quarks. It can be made from a total of six possible quarks, each in three possible colors, and each with either a spin-up or spin-down. From this information, how many possible baryons can be made?

15. A meson is composed of a quark-antiquark pair. It can be made from a total of six possible quarks, each in three possible colors, and each with either a spin-up or spin-down, and six possible antiquarks each in three possible colors, and each with either a spin-up or spin-down. Neglecting linear combinations of these quarks, how many possible mesons can be made?

16. From problems 14 and 15 determine the total number of possible hadrons, ignoring possible mesons made from linear combinations of quarks and antiquarks. Could you make a "periodic table" from this number? Discuss the attempt to attain simplicity in nature.

17. Determine all possible quark combinations that could form a baryon of charge +1 and spin ½.

Bibliography

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Although there are many modern physics textbooks on the market, I would like to list a favorite of mine.

Beiser, Arthur ***Perspectives of Modern Physics.*** New York, McGraw-Hill Book Co., 1969 This is a delightful text on modern physics. I have used it in my modern physics classes since it was first published and its flavor can be seen throughout the modern physics topics in this text.

Popular Physics Books

There are a large number of books on the market today that treat some of the latest topics in the field of physics. These books are very meaningful and very accessible to the student who is taking or has completed a course in physics. The list is divided into two groups; Relativity, and Quantum Physics, although there is an overlap in many of the books, as might be expected.

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